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CAMBRIDGE MATHEMATICAL SERIES.

ELEMENTARY ALGEBRA.

PART II.

GEORGE BELL AND SONS
LONDON: PORTUGAL ST., LINCOLN'S INN
CAMBRIDGE: DEIGHTON, BELL AND CO.
NEW YORK: THE MACMILLAN COMPANY
BOMBAY: A. H. WHEELER AND CO.

ELEMENTARY ALGEBRA

PART II.

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LONDON
GEORGE BELL AND SONS
1905

*** *This volume may be had with or without answers.*

Part II. First published December 1904.
Reprinted January 1905.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS
BY ROBERT MACLEHOSE AND CO. LTD.

PREFACE TO PART II.

THE object of the writers has been to provide a text book of practical interest and utility, fulfilling the latest requirements of the various examining bodies, and following, to a great extent, the recommendations of the Mathematical Association.

Graphical work, involving largely the use of squared paper, is freely employed and interwoven throughout the book. It is used in connection with solution of equations, square and cube roots, statistics, height and distance problems, rate problems of various kinds, indeterminate equations, logarithms, ratio and variation.

The bookwork is expressed in the manner suggested by much experience with learners as the one most readily grasped and retained.

Sets of revision papers are inserted at various stages, usually at the end of what may be considered a term's work.

With a view to practical utility and as a stimulus to interest, logarithms are introduced as early as possible, viz., immediately after Proportion.

Thanks are due to various bodies, from whose examination papers many examples have been taken, especially to the Oxford and Cambridge Local Examination Delegates, and the Controller of His Majesty's Stationery Office.

The method of obtaining logarithms to the base 10, as explained in Article 247, resembles one which was used for some numbers by Napier, after he had adopted that base on the suggestion of Briggs. [See Hutton's *Mathematical Dictionary*, 1796 edition.] It seems to have been re-discovered and developed by Mr. Edser and Professor Perry working independently. The few lines of logarithms and antilogarithms, which are quoted to illustrate the use of tables, are extracted from the tables compiled by Mr Castle.

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PART II.

CHAPTER XXXIV.

184. We will here prove the various Laws employed in Algebra, the truth of which has been assumed in previous chapters.

Commutative Law.

(*a*) *The order in which additions and subtractions are performed is immaterial.*

This is shown to be true in Art. 9, and is usually accepted as self-evident.

(*b*) *The order in which multiplications and divisions are performed is immaterial.*

(1) *For integral numbers.*

$3 \times 7 = 3+3+3+3+3+3+3$, *i.e.* 3 taken 7 times,
$= 21$.

Similarly $7 \times 3 = 7+7+7 = 21$.

Thus we see that if a and b are positive integers,

$$a \times b = b \times a, \text{ or } ab = ba.$$

Similarly, $a \times b \times c = a \times c \times b$ (by the above)
$= c \times a \times b$ and so on.

Again $a \times b \div c$ means that a has to be made b times as great and to be divided into c equal parts. But if a be first divided into c equal parts, and if then each part be made b times as great, the result is the same as before

$$\therefore\ a \times b \div c = (a \div c) \times b.$$

(2) *For Fractions.*

Let a, b, c, d be positive integers.

The expression $\frac{a}{b}$ means that the unit has been divided into b equal parts, of which a parts are taken.

When, therefore, we multiply $\frac{a}{b}$ by $\frac{c}{d}$, we mean that $\frac{a}{b}$ is divided into d equal parts, each equal to $\frac{a}{bd}$, and that c of these parts are taken.

$$\therefore \frac{a}{b} \times \frac{c}{d} = \frac{a}{bd} \times c = \frac{ac}{bd}.$$

It follows by similar reasoning that

$$\frac{a}{b} = \frac{a}{b} \times \frac{c}{c} = \frac{ac}{bc}.$$

Associative Law.

185. *In addition and subtraction, we may group (bracket) the terms of an expression in any way.*

$$a+b-c-d = a+(b-c)-d$$
$$= a-c+(b-d)$$
$$= (a+b)-(c+d) = a-(c-b)-d \text{ and so on (See Art. 14)}$$

In the same way, in multiplication and division,

$$abcd \div e = a \times (bc) \times d \div e$$
$$= \frac{abc}{e} \times d = \frac{a}{e} \times bcd \text{ and so on.}$$

186. The **Law of Indices** is a particular case of the above.

Thus, $a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3}$.

$$a^5 \div a^3 = (aaaaa) \div (aaa) = \frac{aaaaa}{aaa} = aa = a^2 = a^{5-3}.$$

Distributive Law.

187. If a and b have any values whatever, and c is a positive integer, $(a+b)c = (a+b)+(a+b)+\ldots$ taken c times

$$= a+a+a+\ldots \quad \ldots\ldots\ldots\ldots$$
$$+ b+b+b+\ldots \quad \ldots\ldots\ldots\ldots$$
$$= ac+bc. \ldots\ldots\ldots\

$$\therefore\ (a+b)\times c\div d=[(a+b)\times c]\div d$$
$$=(ac+bc)\div d$$
$$=ac\div d+bc\div d,$$

or
$$(a+b)\times\frac{c}{d}=a\times\frac{c}{d}+b\times\frac{c}{d}.$$

$\therefore$ the law illustrated in (1) is shown to be true whether c be a whole number or a fraction, as long as c is positive.

Again, if
$$(a+b)c=ac+bc,$$
$$(a+b)(-c)=-(a+b)c \text{ by the Law of Signs}$$
$$=-(ac+bc)$$
$$=-ac-bc$$
$$=a(-c)+b(-c);$$

$\therefore$ for all values of c $(a+b)c=ac+bc$.

In the case of division,
$$(a+b)\div c=(a+b)\times\frac{1}{c}$$
$$=a\times\frac{1}{c}+b\times\frac{1}{c}$$

where c is any quantity, positive or negative (proved above);
$$\therefore\ (a+b)\div c=a\div c+b\div c.$$

Also
$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}$$
$$=ad\div bd+bc\div bd$$
$$=\frac{ad+bc}{bd}.$$

188. *If an expression has a factor, any multiple of the expression is divisible by that factor.*

Let P be a factor of A, then P will be a factor of mA.

For suppose A $=a$P, then mA $=ma$P.

$\therefore$ P is a factor of mA, *i.e.* mA is divisible by P.

189. *If two expressions have a common factor, the sum and the difference of any multiples of those expressions are divisible by that factor.*

Let P be a common factor of A and B, we have to prove that P is a factor of mA $\pm n$B.

Let $A = aP$, $B = bP$.

Then $mA \pm nB = maP \pm nbP = (ma \pm nb)P$.

$\therefore$ P is a factor of $mA \pm nB$.

190. *To prove the rule for finding the* H.C.F. *of two algebraic expressions.*

Let A, B be the two expressions, from which it is supposed that monomial factors have been removed; and let them be arranged in descending powers of some common letter.

Divide A by B, supposing that A is not of lower degree in that letter than B. Let the quotient be p and remainder C.

Divide B by C. Let the quotient be q and remainder D.

Divide C by D and let the remainder be zero.

The work is shown below.

$$\begin{array}{r|l} B & A \;(p \\ & \underline{pB} \\ \hline C & B\;(q \\ & \underline{qC} \\ \hline D & C\;(r \\ & \underline{rD} \end{array}$$

From this we see that

$A - pB = C$ ..... (1)

$B - qC = D$ ..... (2)

$rD = C$ ..... (3)

To show that D is a common factor of A and B.

From (1) $A = pB + C$

$= p(qC + D) + C$ ... from (2)

$= p(qrD + D) + rD$ ... from (3)

$= (pqr + p + r)D$;

$\therefore$ D is a factor of A.

From (2) $B = qC + D = qrD + D = (qr + 1)D$;

$\therefore$ D is a factor of B.

Thus D is a common factor of A and B.

Next, to show that it is the *highest* common factor.

All the common factors of A and B divide $A - pB$, *i.e.* C.

Thus every common factor of A and B divides $B - qC$, *i.e.* D.

Thus D is a common factor, and all the other common factors divide it;

$\therefore$ D is the H.C.F.

Monomial factors were removed from A and B;

$\therefore$ D contains no such factors.

Moreover, if at any stage a monomial factor were introduced for convenience into *either* the divisor or the dividend (say D or C), it would not appear in the result; for it would not be a common factor of D and C, and consequently not a common factor of A and B.

Also a monomial factor might at any stage be removed without affecting the result; for such factors do not appear in D.

191. *The* L.C.M. *of two expressions is equal to the product of the expressions, divided by their* H.C.F.

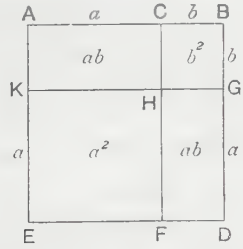
Or, *The* L.C.M. *of two expressions multiplied by their* H.C.F. *is equal to the product of the two expressions.*

Let L be the L.C.M., and H the H.C.F. of A and B.

Let $A = aH$, $B = bH$, so that, by hypothesis *a and b have no common factor.*

$\therefore$ The L.C.M. of A and B is the L.C.M. of aH and bH, which is abH, which $= \frac{AB}{H}$.

192. *Graphical illustration of* $(a+b)^2 = a^2 + 2ab + b^2$.



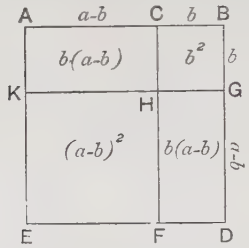
Take a str. line ACB, such that $AC = a$, and $CB = b$ units of length. On AB describe the sq. ABDE, and thro. C draw CF ‖ to AE or BD. From BD cut off BG equal to BC, and thro. G draw GHK ‖ to AB or ED, meeting CF at H.

The figures CG, KF, AH, HD are all rectangles.

Thus
$$\begin{aligned}(a+b)^2 &= \text{fig. ABDE}\\ &= KF + AH + HD + CG\\ &= a^2 + ab + ab + b^2\\ &= a^2 + 2ab + b^2.\end{aligned}$$

193. *Graphical illustration of* $(a-b)^2 = a^2 - 2ab + b^2$,

i.e. $(a-b)^2 + 2ab = a^2 + b^2$.



Take a str. line ACB such that AB $= a$, and CB $= b$ units of length.

Then AC $= a - b$ units.

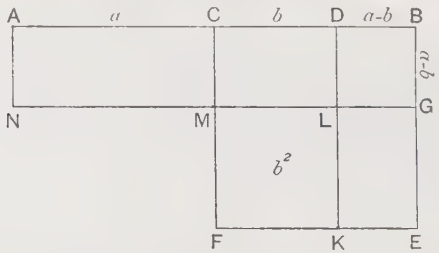
On AB describe the sq. ABDE, and thro. C draw CF ⊥ to AE or BD. From BD cut off BG equal to BC, and thro. G draw GHK ∥ to AB or DE, meeting CF at H.

The figures CG, KF, AH, HD are all rectangles.

Thus
$$\begin{aligned} a^2 &= \text{ABDE} \\ &= \text{KF} + \text{AH} + \text{HD} + \text{CG} \\ &= (a-b)^2 + 2b(a-b) + b^2 \\ &= (a-b)^2 + 2ab - 2b^2 + b^2 \\ &= (a-b)^2 + 2ab - b^2 ; \end{aligned}$$

$\therefore$ adding $b^2 - 2ab$ to both sides, $a^2 - 2ab + b^2 = (a-b)^2$.

194. *Graphical illustration of* $(a+b)(a-b) = a^2 - b^2$.



Take a str. line ACDB, such that AC $=$ CB $= a$, and CD $= b$ units of length.

On CB describe the sq. CBEF, and from BE cut off BG equal to BD. Draw GLMN || to AB or EF, and DLK and AN || to BE or CF

$AD = a + b$, $BD = CB - CD = a - b$.

$\therefore\ (a+b)(a-b) = AD \,.\, DB = AL$

$= AM + CL = DE + CL$ (for $AC = CB = BE$ and $DB = DL$)

$= CBEF - MLKF$

$= a^2 - b^2$.

CHAPTER XXXV.

HARDER FACTORS.

195. *For all positive integral values of n, the expression $x^n - a^n$ is divisible by $x - a$.*

When we divide $x^n - a^n$ by $x - a$, the remainder $= a^n - a^n = 0$. (Remainder Theorem, Art. 95.)

$\therefore\ x^n - a^n$ is divisible by $x - a$, as long as n is integral and positive.

We have already used simple cases of this,

$$x^2 - a^2 = (x-a)(x+a),\quad (x^3 - a^3) = (x-a)(x^2 + ax + a^2).$$

Again, n being a positive integer, the remainder, when we divide $x^n + a^n$ by $x - a$, is $a^n + a^n = 2a^n$; $\therefore\ x^n + a^n$ is never divisible by $x - a$ when n is a positive integer.

If n be a positive **even** *integer, $x^n - a^n$ is divisible by $x + a$.*

When we divide $x^n - a^n$ by $x + a$, the remainder is $(-a)^n - a^n$.

Now if n be an *even* integer $(-a)^n = a^n$;

$\therefore$ the remainder is zero;

$\therefore\ x^n - a^n$ is then divisible by $x + a$.

If n be an *odd* integer, the remainder

$$= (-a)^n - a^n = -a^n - a^n = -2a^n,$$

which is not zero; $\therefore$ when n is an *odd* positive integer, $x^n - a^n$ is not divisible by $x + a$.

196. *If n be any* **odd** *positive integer, the expression $x^n + a^n$ is divisible by $x + a$.*

The remainder, when we divide $x^n + a^n$ by $x + a$, is $(-a)^n + a^n$.

Now if n be an *odd* positive integer, $(-a)^n = -a^n$;

$\therefore$ the remainder is zero;

i.e. $x^n + a^n$ is then divisible by $x + a$.

If n be an *even* positive integer, the remainder $= a^n + a^n = 2a^n$, which is not zero; $\therefore$ $x^n + a^n$ is not divisible by $x + a$ in that case.

197. The theory of quadratic equations is often useful in the determination of factors.

Example 1. Consider the expression $3x^2 - 5x + 7$.

If we solve the equation $3x^2 - 5x + 7 = 0$,

the quantity under the radical sign $= 25 - 4 \times 3 \times 7$ $(b^2 - 4ac)$

$= 25 - 84$,

which is negative; $\therefore$ the equation has imaginary roots;

$\therefore$ the expression $3x^2 - 5x + 7$ has no real factors.

Example 2. Find factors of the expression $x^2 - 4x + 1$.

If we solve the equation $x^2 - 4x + 1 = 0$,

$$x = \frac{4 \pm \sqrt{16 - 4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3};$$

$\therefore$ the reqd. factors are $x - 2 - \sqrt{3}$ and $x - 2 + \sqrt{3}$.

The method of finding factors by completing squares is useful.

$$x^2 - 4x + 1$$
$$= x^2 - 4x + 4 - 3$$
$$= (x - 2)^2 - 3$$
$$= (x - 2 + \sqrt{3})(x - 2 - \sqrt{3}).$$

198. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

This can be seen by dividing $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$. (See Ex. 2, Art. 93.)

By writing $-b$ instead of b in the above we see that

$$a^3 - b^3 + c^3 + 3abc = (a - b + c)(a^2 + b^2 + c^2 + ab + bc - ca).$$

In the same way, writing $-a$ for a,

$$-a^3 + b^3 + c^3 + 3abc = (-a + b + c)(a^2 + b^2 + c^2 + ab - bc + ca)$$

and similarly

$$a^3 + b^3 - c^3 + 3abc = (a + b - c)(a^2 + b^2 + c^2 - ab + bc + ca).$$

Example 1. $x^3 + 8y^3 + z^3 - 6xyz$

$= x^3 + (2y)^3 + z^3 - 3(2x)yz$

$= (x + 2y + z)(x^2 + 4y^2 + z^2 - 2xy - 2yz - zx)$.

Example 2. Factorize the expression

$$(x^2-3x+4)^2-5x(x^2-3x+4)+6x^2.$$

The expression is in the form $a^2-5ax+6x^2$ which is equal to

$$(a-2x)(a-3x);$$

$$\therefore \text{ the expression} = (x^2-3x+4-2x)(x^2-3x+4-3x)$$
$$= (x^2-5x+4)(x^2-6x+4)$$
$$= (x-1)(x-4)(x^2-6x+4).$$

Example 3. Factorize the expression $a^4+a^2b^2+b^4$.

$$a^4+a^2b^2+b^4 = (a^4+2a^2b^2+b^4)-a^2b^2$$
$$= (a^2+b^2)^2-a^2b^2$$
$$= (a^2+b^2+ab)(a^2+b^2-ab)$$
$$= (a^2+ab+b^2)(a^2-ab+b^2).$$

Example 4. Factorize the expression

$$x^4-5x^2-12-\frac{5}{x^2}+\frac{1}{x^4}.$$

$$\text{The given expression} = x^4+\frac{1}{x^4}-5\left(x^2+\frac{1}{x^2}\right)-12$$
$$= \left(x^2+\frac{1}{x^2}\right)^2-2-5\left(x^2+\frac{1}{x^2}\right)-12$$
$$= \left(x^2+\frac{1}{x^2}\right)^2-5\left(x^2+\frac{1}{x^2}\right)-14$$
$$= a^2-5a-14, \text{ writing } a \text{ for } x^2+\frac{1}{x^2}$$
$$= (a-7)(a+2)$$
$$= \left(x^2-7+\frac{1}{x^2}\right)\left(x^2+2+\frac{1}{x^2}\right)$$
$$= \left(x^2-7+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)^2.$$

Example 5. Factorize the expression

$$6x^2+7xy-3y^2-14x-10y+8.$$

Consider the equation $6x^2+7xy-3y^2-14x-10y+8=0$.
Arranging as a quadratic for x,

$$6x^2+x(7y-14)-3y^2-10y+8=0.$$

$$x=\frac{-(7y-14)\pm\sqrt{(7y-14)^2-24(-3y^2-10y+8)}}{12}, \quad \left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{-7y+14\pm\sqrt{121y^2+44y+4}}{12}$$

$$=\frac{-7y+14\pm(11y+2)}{12}$$

$$=\frac{4y+16}{12} \text{ or } \frac{-18y+12}{12}$$

$$=\frac{y+4}{3} \text{ or } \frac{-3y+2}{2}.$$

$\therefore$ $3x-y-4$ and $2x+3y-2$ are the reqd. factors.

DIVISION BY MEANS OF FACTORIZATION.

199. *Divide* $x^2+2(a+b)x+a^2+2ab+b^2$ *by* $x+a+b$.

$$x^2+2x(a+b)+a^2+2ab+b^2=x^2+2x(a+b)+(a^2+2ab+b^2)$$
$$=(x+\overline{a+b})^2. \quad [(p+q)^2=p^2+2pq+q^2]$$

$\therefore$ $x+a+b$ is the reqd. quotient.

Divide $a^2+b^2-c^2-d^2+2ab-2cd$ *by* $a+b-c-d$.

$$a^2+b^2-c^2-d^2+2ab-2cd=(a^2+2ab+b^2)-(c^2+2cd+d^2)$$
$$=(a+b)^2-(c+d)^2$$
$$=(a+b+c+d)(a+b-c-d).$$

$\therefore$ $a+b+c+d$ is the reqd. quotient.

Divide $x^2+2xy+y^2+x+y-6$ *by* $x+y-2$.

$$x^2+2xy+y^2+x+y-6=(x+y)^2+(x+y)-6$$
$$=(x+y+3)(x+y-2).$$
$$[a^2+a-6=\overline{a+3}\,.\,\overline{a-2}]$$

$\therefore$ $x+y+3$ is the reqd. quotient.

Examples. XXXV. a.

1. Prove, without actual division, that x^6-a^6 is divisible by $x-a$ without remainder. Find three terms of the quotient by division, and hence write down the complete quotient.

Do the same with

2. x^7+a^7 and $x+a$.
3. x^7-a^7 and $x-a$.
4. x^8-a^8 and $x+a$.
5. x^8-a^8 and $x-a$.
6. x^6-64 and $x-2$.
7. x^6-64 and $x+2$.

Factorize the following expressions:

8. x^3+x^2-4x-4.
9. $(x^2+x)^2-2(x^2+x)-3$.
10. $\frac{27}{z^3}-1$.
11. $a^3+\frac{1}{a^3}$.
12. $(x^2-4x)^2+7(x^2-4x)+12$.
13. $x^2y^2-x^2-y^2+1$.
14. $c(c-b)+a(b-a)$.
15. x^2-2x-2.
16. a^4-23a^2+1.
17. $abx^3+(a^2-b)x^2-1$.
18. $a(a+c)-b(b+c)$.
19. $a^3+b^3+1-3ab$.
20. $x^2-10x+23$.
21. $4x^2-4xy+y^2-10x+5y-24$.
22. x^4+4.
23. $x^3-8y^3-27-18xy$.
24. $4x^2-8x+1$.
25. $9x^2-16y^2-9x+28y-10$.
26. $15x^2-16xy+4y^2+41x-22y+28$.
27. x^4+x^2+1.
28. $a^3-b^3+8+6ab$.
29. $16x^2+8x-5$.
30. $16x^4+4x^2+1$.
31. $8a^3+b^3-1+6ab$.
32. $(2x^2-5x+3)(2x^2-5x+4)-2$.
33. $6x^2-19xy+15y^2+23x-36y+21$.
34. x^4+64.
35. $x^4-x^2y^2+16y^4$.
36. $(x^2+2x-8)(x^2+2x-2)-27$.
37. $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$.
38. $81a^4+9a^2+1$.
39. $2x^2+5xy+3y^2+6x+10y-8$.
40. x^4+4x^2+16.
41. $81x^4+64$.
42. $x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5$.
43. $x^2+x-10+\frac{1}{x}+\frac{1}{x^2}$.
44. $x^3+11x-6(x^2+1)$.
45. $x^2-5x-10-\frac{10}{x}+\frac{4}{x^2}$.
46. $2x^3+5x^2+x-2$.
47. $x^3-5x-\frac{5}{x}+\frac{1}{x^3}$.
48. $x^4+3x^2-8+\frac{3}{x^2}+\frac{1}{x^4}$.

[*The following examples should be done by using your knowledge of factors, and not by actual multiplication and division.*]

Find the product of:

49. $x+2y+3z$ and $x+2y-3z$.
50. $2x-3y+4z$ and $2x+3y+4z$.
51. $a+b-2$ and $a+b-3$.
52. $a+b+x+y$ and $a+b-x-y$.
53. x^2-x+1 and x^2+x+1.
54. x^2-x+2 and x^2+x+2.
55. x^2-x-2 and x^2+x+2.
56. $x^2-3xy+5y^2$ and $x^2+3xy+5y^2$.
57. $2a+b+x+2y$ and $2a+b-x-2y$.
58. $2x^2+3xy-4y^2$ and $2x^2-3xy+4y^2$.
59. x^3+x^2+x+1 and x^3+x^2-x-1.
60. x^2-2x+4 and x^2+2x+4.
61. $(2a+b)^2-(a+2b)^2$ and $(2a-b)^2-(a-2b)^2$.
62. $x+1+\frac{1}{x}$ and $x-1+\frac{1}{x}$.
63. x^3-2x^2+x-2 and x^3+2x^2+x+2.
64. $(x^2+x+1)^2-(x^2-x+1)^2$ and $(2x^2+x-2)^2-(2x^2-x-2)^2$.

65. $2ax-2x^2$, $3ax+3x^2$, $a^4+a^2x^2$.
66. $(a-b)$, $(a+b)$, (a^2+ab+b^2), (a^2-ab+b^2).
67. $(a-b)^2$, $(a+b)^2$, $(a^2+b^2)^2$.
68. $(3x-2)$, $(5x+4)$, $(15x^2-2x-8)$.
69. $(6x^2+5x-4)$, $(3x-4)$, $(2x+1)$.
70. $(x-2)$, $(x+2)$, (x^2-2x+4), (x^2+2x+4).
71. $(a-b)^3$, $(a+b)^3$, $(a^2+b^2)^3$.
72. $(3a+b)^2-(a+3b)^2$, $(3a-b)^2-(a-3b)^2$, $(a^2+b^2)^2$.
73. $(a+b+c)$, $(a+b-c)$, $(a-b+c)$, $(a-b-c)$.

74. Divide $(3x^2+2x+1)^2-(2x^2-x+5)^2$ by x^2+3x-4.
75. Simplify $(6x^2+x-1)\times(2x^2-9x+4)\div(3x^2-13x+4)$.
76. Show that $(2x^2+3x+2)^3-(x^2+x+1)^3$ is divisible by $(x+1)^2$.
77. Divide the product of $2x^2+x-6$ and $3x^2-5x-2$ by $2x^2-7x+6$.
78. Divide $(a-2b+3)^2-(a+2b-5)^2$ by $a-1$.
79. Show that $(3x^2-7x+5)^3+(x^2+3x-4)^3$ is divisible by $(2x-1)^2$.
80. Simplify $[x^2+ax-bx-ab]\times[x^2-ax+bx-ab]\div[x^2-a^2]$.
81. Multiply the square of $3a-2b$ by $9a^2+12ab+4b^2$.

Write down the quotient when

82. x^4-y^4 is divided by $x-y$.
83. x^4-y^4 is divided by $x+y$.
84. x^4-16y^4 is divided by $x-2y$.
85. x^4-16y^4 is divided by $x+2y$.
86. Prove that $x^{140}-a^{140}$ is divisible by $x-a$, by x^2-a^2, and by x^2+a^2.
87. Prove that $x^6+3x^4+4x^2+224$ is divisible by x^2+7.

SYMMETRICAL EXPRESSIONS.

200. An expression is said to be *symmetrical* when it is unaltered by interchanging *any* pair of the letters which it contains.

Thus the following are symmetrical expressions:

$$a+b+c, \quad a^3+b^3+c^3,$$

$$a^2+b^2+c^2-ab-bc-ca, \quad ab+bc+cd+da+ac+bd.$$

The expression $(a-b)(b-c)(c-a)$ is not symmetrical, for if we interchange a and b, the expression becomes

$$(b-a)(a-c)(c-b),$$

$$i.e. \quad -(a-b)(c-a)(b-c).$$

In connection with symmetrical expressions the following notation is useful.

$\sum_{abc}[a(b-c)]$ = the sum of all terms like $a(b-c)$ when we interchange the letters a, b, c, in order;

i.e. $$\sum_{abc}[a(b-c)] = a(b-c)+b(c-a)+c(a-b);$$

$$\sum_{abc}\frac{a^2}{(a-b)(a-c)} = \frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-c)(b-a)}+\frac{c^2}{(c-a)(c-b)}.$$

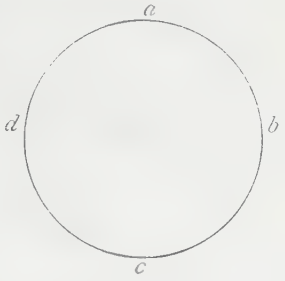
The following are simpler cases:

$$\sum_{abc}(a^2) = a^2+b^2+c^2;$$

$$\sum_{abc}(a^2-bc) = (a^2-bc)+(b^2-ca)+(c^2-ab).$$

CYCLIC SYMMETRY. CYCLIC ORDER.

201. When an expression is written so that the letters forming it are taken in a selected unending order (generally the order of the alphabet) it is said to be in *cyclic symmetry*, and the letters are said to be taken in *cyclic order*.



This is well illustrated by arranging the letters in order round the circumference of a circle.

Arranging the letters a, b, c, d in this way and looking at the expression

$$(a-b)(b-c)(c-d)(d-a)$$

we see that the letters are taken in cyclic order, and the expression has cyclic symmetry.

On the other hand, in the expression

$$(a-b)(c-b)(d-c)(d-a)$$

the order is changed in the second and third factors.

By changing the signs in these factors, the expression may be written $$(a-b)(b-c)(c-d)(d-a),$$

and is now reduced to cyclic symmetry.

The work will be much simplified if the expressions used are first arranged in cyclic symmetry, where such is possible.

Dimensions. The **dimensions** of any algebraical term are determined by taking the sum of the indices of the various letters employed.

Thus $9x^2y^3$ is of five dimensions; $17x^3yz^4$ is of eight dimensions. In the latter case it must be remembered that the index of y is unity.

Numerical coefficients are not taken into account.

Homogeneous. A compound algebraical expression is said to be *homogeneous* when all its terms are of the same dimensions. $x^2+2xy+y^2$ is homogeneous, every term being of two dimensions. $a^5-5a^4b+6a^3b^2-5a^2b^3+ab^4-b^5$ is homogeneous, every term being of five dimensions.

HOMOGENEOUS AND SYMMETRICAL EXPRESSIONS.

202. *Find the factors of* $ab(a-b)+bc(b-c)+ca(c-a)$.

Let X denote the given expression.

When $a=b$,

$$\begin{aligned} X &= 0+bc(b-c)+cb(c-b),\\ &= bc(b-c+c-b)\\ &= 0. \end{aligned}$$

$\therefore$ $a-b$ is a factor of X, by the Remainder Theorem.

In the same way we may prove that $b-c$ and $c-a$ are also factors of X.

Also no two of the factors $a-b$, $b-c$, $c-a$ have a common factor,

$\therefore$ X is divisible by $(a-b)(b-c)(c-a)$ without remainder

Again, X is entirely of the third degree,

and $(a-b)(b-c)(c-a)$ is also of the third degree,

$\therefore$ the only other possible factor of X is a numerical one

Let that factor be k, so that

$$ab(a-b)+bc(b-c)+ca(c-a)=k(a-b)(b-c)(c-a).$$

This equation must be true for all values of a, b, and c,

$\therefore$ putting $a=0$, $b=1$, and $c=-1$,

$$-1(1+1)=k(-1)(2)(-1),$$

$$k=-1.$$

$\therefore$ $ab(a-b)+bc(b-c)+ca(a-c)=-(a-b)(b-c)(c-a)$.

The value of k might be determined in the following manner

Examine the equation

$$ab(a-b)+bc(b-c)+ca(c-a)\equiv k(a-b)(b-c)(c-a).$$

The coeff. of a^2b on the *left-hand side* is unity.

The coeff. of a^2b on the *right-hand side* is $-k$.

(We see this by taking the product of k, with a in the first factor, b in the second factor, and $-a$ in the third factor.)

$$\therefore \quad -k=1 \quad \text{or } k=-1.$$

The expression $ab(a-b)+bc(b-c)+ca(c-a)$ may be written in any of the following forms

$$a^2b-b^2a+b^2c-c^2b+c^2a-a^2c,$$

$$a(c^2-b^2)+b(a^2-c^2)+c(b^2-a^2),$$

or
$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

We therefore have the following important results:

$$\begin{aligned}
&\mathbf{ab(a-b)+bc(b-c)+ca(c-a)}\\
&\equiv \mathbf{a^2b-b^2a+b^2c-c^2b+c^2a-a^2c}\\
&\equiv \mathbf{a(c^2-b^2)+b(a^2-c^2)+c(b^2-a^2)}\\
&\equiv \mathbf{a^2(b-c)+b^2(c-a)+c^2(a-b)}\\
&\equiv \mathbf{-(a-b)(b-c)(c-a)},
\end{aligned}$$

or

$$\sum_{abc}[ab(a-b)] \equiv \sum_{abc}[a(c^2-b^2)] \equiv \sum_{abc}[a^2(b-c)] \equiv -(a-b)(b-c)(c-a)$$

203. *Factorize the expression* $\sum_{abc}[a^3(b-c)]$,

i.e. $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

Let X denote the given expression.

When
$$b=c,$$
$$X=b^3(b-a)+b^3(a-b)=0;$$
$$\therefore \ b-c \text{ is a factor of X.}$$

In the same way it may be proved that $c-a$ and $a-b$ are also factors.

Now X is of the *fourth* degree and $(a-b)(b-c)(c-a)$ is of the *third* degree.

$\therefore$ there will be one more factor, and it must be of the *first* degree.

Also it must be *symmetrical* with regard to a, b, and c.

It must therefore be $k(a+b+c)$ where k is some numerical quantity.

$$\therefore \ a^3(b-c)+b^3(c-a)+c^3(a-b)\equiv k(a-b)(b-c)(c-a)(a+b+c) \ldots (1)$$

Putting $a=0,\ b=1,\ c=2,$

$$2+8(-1)=k(-1)(-1)(2)(3)$$

whence $$k=-1;$$

$$\therefore\ \sum_{abc}[a^3(b-c)]\equiv-(a-b)(b-c)(c-a)(a+b+c).$$

k might be determined as follows.

The coefft. of a^3b on the left in (1) is unity.

..... right ..... is $-k$.

(This is found by taking a from the first and fourth factors, $-a$ from the third, and b from the second.)

$$\therefore\ -k=1,\ \textit{i.e.}\ k=-1.$$

204. *Find the factors of*

$$\sum_{abc}[a^2(b-c)^3],\ \textit{i.e.}\ a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3.$$

Let X denote the given expression.

When $a-b=0,$ $\quad X=a^2(a-c)^3+a^2(c-a)^3$

$=0$, for $(c-a)^3=[-(a-c)]^3=-(a-c)^3$.

$\therefore\ a-b$ is a factor of X.

In the same way it may be proved that

$b-c$ and $c-a$ are factors.

$\therefore$ X is divisible by $(a-b)(b-c)(c-a)$.

Now this expression is of the *third* degree, and X is of the *fifth* degree, therefore the remaining factor must be of the *second* degree.

Moreover it must be *symmetrical* with regard to a, b, and c.

$\therefore$ it must be of the form

$$k(a^2+b^2+c^2)+l(ab+bc+ac),$$

where k and l are numerical quantities.

$$\therefore\ a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$$

$$\equiv(a-b)(b-c)(c-a)[k(a^2+b^2+c^2)+l(ab+bc+ca)].$$

When $a=0,\ b=1,\ c=2,$

$$(2)^3+4(-1)^3=(-1)(-1)(2)[k(1+4)+l(2)]$$

$$\textit{i.e.}\ 5k+2l=2. \qquad \ldots\ldots(1)$$

When $a=0,\ b=-1,\ c=2,$

$$(2)^3+4(1)^3=(1)(-3)(2)[k(1+4)+l(-2)]$$

$$\textit{i.e.}\ 5k-2l=-2. \qquad \ldots\ldots(2)$$

To find k and l we must solve equations (1) and (2).

By addition $k=0$.

By subtraction $l=1$.

$$\therefore \sum_{abc}[a^2(b-c)^3]=(a-b)(b-c)(c-a)(ab+bc+ca).$$

Methods of testing accuracy of working—

(1) In dealing with homogeneous expressions it is useful to notice that if two such expressions are multiplied together, or if one is divided by the other, the result must be homogeneous.

Thus if an expression in which every term is of the 5^{th} degree be multiplied by one in which every term is of the 2^{nd} degree, the product must be homogeneous of the 7^{th} degree. Any term which does not stand this test betrays an error.

Similarly, if the former expression be divided by the latter, the quotient must have all its terms of the 3^{rd} degree.

(2) Results may be checked by substituting special values for the letters employed.

(3) In multiplying together symmetrical expressions the product must be symmetrical.

205. The equation $x^2+y^2=0$ may be interpreted geometrically. We know that the equation $x^2+y^2=c^2$ represents a circle whose centre is at the point (0, 0) and whose radius $=c$.

By making c zero we see that the equation $x^2+y^2=0$ represents a circle whose centre is at the point (0, 0) and whose radius $=0$, *i.e.* it represents the points (0, 0).

$\therefore$ the equation $x^2+y^2=0$ is equivalent to the statement that $x=0$ *and* $y=0$.

From this we see that, if an equation can be reduced to the statement that the sum of certain squares is zero, we can equate these squares simultaneously to zero.

Examples. XXXV. b.

Write down in full:

1. $\sum_{abc}(a^3-bc)$.
2. $\sum_{xyz}\frac{yz}{(y+x)(z+x)}$.
3. $\sum_{abc}\frac{a^3-bc}{(b-a)(c-a)}$.
4. $\sum_{abc}\frac{x^2-a^2}{(a-b)(a-c)}$.
5. $\sum_{abc}[(a-b)(a-c)(x+a)]$.

Find the factors of the following expressions:

6. $\sum_{abc}(a-b)^3$.
7. $\sum_{abc}[a^2(b^3-c^3)]$.
8. $\sum_{xyz}[x^4(y^2-z^2)]$.
9. $\sum_{abc}[bc(b^3-c^3)]$.

Prove the following identities:

10. $\sum_{abc}[a^2b^2(a-b)]\equiv-(a-b)(b-c)(c-a)(ab+bc+ca)$.
11. $\sum_{xyz}[x^4(y-z)]\equiv-(x-y)(y-z)(z-x)(x^2+y^2+z^2+xy+yz+zx)$.
12. $\sum_{abc}[a(b-c)^3]\equiv(a-b)(b-c)(c-a)(a+b+c)$.
13. $\sum_{abc}(a-b)^5\equiv5(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$.
14. $\sum_{abc}[(b-c)(x-b)(x-c)]=-(a-b)(b-c)(c-a)$.
15. $(x-y)(x+y)^2+(y-z)(y+z)^2+(z-x)(z+x)^2+(x-y)(y-z)(z-x)\equiv0$.
16. $(a+b+c)^3+(a-b-c)^3+(b-c-a)^3+(c-a-b)^3\equiv24abc$.
17. $(x-y)(x+y)^3+(y-z)(y+z)^3+(z-x)(z+x)^3$
$+2(x-y)(y-z)(z-x)(x+y+z)\equiv0$.
18. $(x+y+z)^5+(x-y-z)^5+(y-z-x)^5+(z-x-y)^5\equiv80xyz(x^2+y^2+z^2)$

Reduce the following to their simplest forms:

(*Check your results by giving a, b, c, etc., particular values.*)

19. $\frac{a+b}{ab+c^2-ac-bc}+\frac{a+c}{ac+b^2-ab-bc}+\frac{b+c}{bc+a^2-ab-ac}$.
20. $\frac{c}{ab(b-c)(c-a)}+\frac{b}{ac(a-b)(b-c)}+\frac{a}{bc(a-b)(c-a)}$.
21. $\frac{x(y+z)}{(x-y)(z-x)}+\frac{y(z+x)}{(x-y)(y-z)}+\frac{z(x+y)}{(z-x)(y-z)}$.
22. $\frac{a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3}{bc+ca+ab}$.
23. $\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}$.
24. $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-z)(y-x)}+\frac{1}{z(z-x)(z-y)}$.
25. $\frac{x^2}{(a-x)(x-y)(x-z)}+\frac{y^2}{(a-y)(y-x)(y-z)}+\frac{z^2}{(a-z)(z-x)(z-y)}$.

26. $\dfrac{2(b-c)(c-a)(a-b)}{(b+c)(c+a)(a+b)} - \dfrac{b^2-c^2}{(c+a)(a+b)} - \dfrac{c^2-a^2}{(a+b)(b+c)} - \dfrac{a^2-b^2}{(b+c)(c+a)}$.

27. $\dfrac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

28. $\sum_{abc} \dfrac{1}{(a-b)(a-c)}$.

29. $\sum_{abc} \dfrac{1}{a(a-b)(a-c)}$.

30. $\sum_{abc} \dfrac{b+c}{(c-a)(a-b)}$.

31. $\sum_{abc} \dfrac{1}{a^2(a^2-b^2)(a^2-c^2)}$.

32. $\sum_{abc} \dfrac{a^2}{(a-b)(a-c)}$.

33. $\sum_{abc} \dfrac{a^3}{(c-a)(a-b)}$.

34. $\sum_{abc} \dfrac{1}{(a-b)(a-c)(x-a)}$.

35. $\sum_{abc} \dfrac{a}{(a-b)(a-c)(x-a)}$.

36. $\sum_{abc} \dfrac{a^2}{(a-b)(a-c)(x+a)}$.

37. $\sum_{abc} \dfrac{a(b+c)}{(a-b)(a-c)}$.

38. $\sum_{abc} \dfrac{a^2-bc}{(a+b)(a+c)}$.

39. $\sum_{xyz} \dfrac{x^2y^2}{(z^2-x^2)(z^2-y^2)}$.

206. *Find the value of λ if the expression $x^2 - xy - 6y^2 + 7x - y + \lambda$ can be resolved into two factors of the first degree.*

Taking the terms of the second degree $x^2 - xy - 6y^2$, we see that they can be resolved into $(x-3y)(x+2y)$.

$\therefore$ the factors of the given expression will be of the form

$$(x - 3y + a)(x + 2y + b).$$

Multiplying, this expression is equal to

$$x^2 - xy - 6y^2 + (a+b)x + (2a - 3b)y + ab.$$

Hence by comparison with the given expression we see that

$$a + b = 7, \quad 2a - 3b = -1, \quad \text{and} \quad ab = \lambda.$$

From the first two of these equations

$$a = 4, \quad \text{and} \quad b = 3, \quad \therefore \quad \lambda = ab = 12.$$

207. *If the expression $xy + gx + fy + c$ can be resolved into two factors of the first degree, find the relation between g, f, and c.*

If the expression has factors they must evidently be of the form $(x+\alpha)$, $(y+\beta)$.

$\therefore$ their product $xy + \beta x + \alpha y + \alpha\beta$ must be the same as the given expression.

$\therefore$ by comparison,

$$\beta = g, \quad \alpha = f, \quad \text{and} \quad \alpha\beta = c.$$

$$\therefore \quad fg = c.$$

208. *If two expressions of the n^{th} degree (n being a positive integer) are identically equal to one another, the coefficient of any power of x in one expression is equal to the coefficient of the same power of x in the other.*

Let the two expressions be

$$ax^n + bx^{n-1} + \ldots + qx^2 + rx + s,$$

and $$a'x^n + b'x^{n-1} + \ldots + q'x^2 + r's + s'.$$

They are equal to one another for all values of x.

$\therefore$ putting x equal to zero, $s = s'$.

$\therefore\ ax^n + bx^{n-1} + \ldots + qx^2 + rx \equiv a'x^n + b'x^{n-1} + \ldots + q'x^2 + r'x.$

Dividing both sides by x,

$$ax^{n-1} + bx^{n-2} + \ldots + qx + r \equiv a'x^{n-1} + b'x^{n-2} + \ldots + q'x + r'.$$

Putting $x = 0$ in this identity,

$$r = r'.$$

In the same way we can prove that $q = q' \ldots b = b'$, and $a = a'$.

209. *To find the relations between the coefficients in order that the expression $x^4 + ax^3 + bx^2 + cx + d$ may be a perfect square.*

If it is a square it must be of the form $(x^2 + px + \sqrt{d})^2$.

$\therefore\ x^4 + ax^3 + bx^2 + cx + d \equiv (x^2 + px + \sqrt{d})^2$

$$\equiv x^4 + 2px^3 + (2\sqrt{d} + p^2)x^2 + 2px\sqrt{d} + d.$$

$\therefore$ by equating coefficients of like powers of x we get

$$2p = a, \quad \ldots\ldots(1)$$

$$2\sqrt{d} + p^2 = b, \quad \ldots\ldots(2)$$

$$2p\sqrt{d} = c. \quad \ldots\ldots(3)$$

$\therefore$ from (1) and (3) by divison $\sqrt{d} = \dfrac{c}{a}$.

$$\therefore\ a^2 d = c^2. \quad \ldots\ldots(4)$$

From (2) $\quad 8\sqrt{d} = 4b - 4p^2$

$$= 4b - a^2 \ldots \text{from (1)}.$$

$\therefore$ by squaring we get

$$64d = (4b - a^2)^2. \quad \ldots\ldots(5)$$

Equations (4) and (5) are the required conditions.

210. *If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$, $\alpha+\beta+\gamma=-p$, $\alpha\beta+\beta\gamma+\gamma\alpha=q$, $\alpha\beta\gamma=-r$.*

α, β, γ evidently satisfy the equation $(x-\alpha)(x-\beta)(x-\gamma)=0$. This may be written

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0. \quad \ldots\ldots\ldots(1)$$

Thus we see that the given equation and equation (1) must be identically the same.

$\therefore$ by comparison

$$\alpha+\beta+\gamma=-p, \quad \alpha\beta+\beta\gamma+\gamma\alpha=q, \quad \text{and} \quad \alpha\beta\gamma=-r.$$

Similar properties may be proved for equations of higher degrees.

The sum of the squares of the roots

$$=(\alpha+\beta+\gamma)^2-2(\beta\gamma+\gamma\alpha+\alpha\beta)=p^2-2q.$$

211. *Find values (different from a and b) for* A *and* B, *which will render the equation $m(x+\mathsf{A})^2+n(x+\mathsf{B})^2\equiv m(x+a)^2+n(x+b)^2$ an identity for all values of x.*

Multiplying out

$$2mx\mathsf{A}+m\mathsf{A}^2+2nx\mathsf{B}+n\mathsf{B}^2\equiv 2mxa+ma^2+2nxb+nb^2.$$

Since this is an identity, the coefficients of x on either side are equal, and the constant terms are equal.

$$\therefore \quad m\mathsf{A}+n\mathsf{B}=ma+nb \quad \ldots\ldots\ldots(1)$$

and
$$m\mathsf{A}^2+n\mathsf{B}^2=ma^2+nb^2.$$

These may be written $m(\mathsf{A}-a)=-n(\mathsf{B}-b)$

$$m(\mathsf{A}^2-a^2)=-n(\mathsf{B}^2-b^2).$$

Dividing and neglecting the solution $\mathsf{A}=a$, $\mathsf{B}=b$

$$\mathsf{A}+a=\mathsf{B}+b.$$

Multiplying this by m and subtracting from (1)

$$\mathsf{B}=\frac{2am+bn-bm}{m+n},$$

and hence
$$\mathsf{A}=\frac{2bn+am-an}{m+n}.$$

Examples. XXXV. c.

1. Find the condition that $x^2 - 8ax + p$ may be a perfect square for all values of x.

2. Prove that $x^n - 1$ is not divisible by $x + a$, unless a is equal to ± 1.

3. Prove that $x^{2n+1} + 1$ is not divisible by $x + a$, unless a is equal to unity.

4. What number must be added to $x^3 - 5x^2$ to make the expression exactly divisible by $x - 1$?

5. If $x^2 - 5px + q^2$ is exactly divisible by $x - p$, prove that $q = \pm 2p$.

6. Find the relation between p, q, and r, if $x^3 - px^2 + qx + r$ is exactly divisible by $x - p$.

7. If $a + b + c = 0$, prove that $a^3 + b^3 + c^3 = 3abc$.

8. Prove that $\sum_{xyz} (x-y)^3 = 3(x-y)(y-z)(z-x)$.

9. Prove that $4px^3 - 8p^2x^2 + 3pq^2x + p^2q^2$ is exactly divisible by $x - p$, if $q = \pm p$.

10. Use the remainder theorem to prove that $\sum_{abc} (b+c-2a)^2 - 9\sum_{abc} (a^2)$ is exactly divisible by $a + b + c$.

11. Solve the equation $(x-a)^2 + (y-b)^2 = 0$. Distinguish between it and the equation $(x-a)(y-b) = 0$.

12. Prove that $\sum_{abc} (b+c-2a)^3 \equiv 3(b+c-2a)(c+a-2b)(a+b-2c)$.

13. Prove that if $\sum_{abc} (a^2) = \sum_{abc} (bc)$, then $a = b = c$.

14. Find the condition that $ax - p$ may be a factor of $ax^2 + bx + c$.

15. Prove that the equation $(ax - by + 2ab)^2 + (ax + by)^2 = 0$ has only one solution, and find that solution.

16. What value of x will make $x^4 - 10x^3 + 29x^2 - 16x - 8$ a perfect square?

17. What must be added to $x^4 - 6x^3 + 17x^2 - 32x + 12$ in order that it may be a perfect square for all values of x?

18. Prove that $(3a-b)^3 + (3b-c)^3 + (3c-a)^3 = 3(3a-b)(3b-c)(3c-a)$ if $a + b + c = 0$.

19. Prove that $\frac{p^2}{3} q - 3r^2$, if $x^3 + px^2 + qx + r^3$ is a perfect cube.

20. Prove that $(x+y+z)^7 - x^7 - y^7 - z^7$ is divisible by $(x+y)(y+z)(z+x)$.

21. Prove that the equation $8x^2 - 4xy + y^2 - 4x + 1 = 0$ has only one solution, and find that solution.

22. Prove that $a\left(\frac{b^3-c^3}{b-c}\right) + b\left(\frac{c^3-a^3}{c-a}\right) + c\left(\frac{a^3-b^3}{a-b}\right) = 0$, if $a + b + c = 0$.

23. Find an integral value of x which will make $(x+1)^2$ greater than $5x - 1$ and less than $7x - 3$.

24. Prove that if $x = a + d$, $y = b + d$, $z = c + d$, then

$$x^2 + y^2 + z^2 - yz - xz - xy = a^2 + b^2 + c^2 - bc - ca - ab.$$

25. Prove that if $\left(\frac{y}{x}+\frac{b}{a}\right)\left(\frac{z}{x}+\frac{c}{a}\right)=\frac{bc}{a^2}$, then $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$.

26. Resolve the following expression into two factors,

$$4x^6-4x^4y^2+12x^3y+x^2y^4-6xy^3+9y^2.$$

27. Find what number must be added to the expression

$$\frac{x^2}{9}+\frac{1}{4x^2}+\frac{2x}{3}+\frac{1}{x}$$ to make it a perfect square.

28. Simplify the expression

$$(b-c)(x-a)^2+(c-a)(x-b)^2+(a-b)(x-c)^2+(b-c)(c-a)(a-b).$$

29. Simplify the following $\frac{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)}{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}$.

30. If $(b+c)x=a$, $(c+a)y=b$, $(a+b)z=c$, prove that

$$yz+zx+xy+2xyz=1.$$

31. If S_n stands for $x^n+\frac{1}{x^n}$, prove that

(i) $S_1^2=S_2+2$. (ii) $S_n^2=S_{2n}+2$.

(iii) $S_1S_2=S_3+S_1$. (iv) $S_nS_1=S_{n+1}+S_{n-1}$.

CHAPTER XXXVI.

SURDS.

212. The n^{th} root of a number which is not a perfect n^{th} power is called a **surd** or **irrational quantity.**

A quadratic surd is the square root of a number which is not a perfect square. $\sqrt{7}$ is a *surd*, $\sqrt{9}$ is a *rational* quantity. Surds may often be simplified by being factorised. Thus

$$\sqrt{147}=\sqrt{49}\,.\,\sqrt{3}=7\sqrt{3}.$$

Conversion into an entire surd would be the reverse of this process. $2\sqrt{3}=\sqrt{4}\,.\,\sqrt{3}=\sqrt{12}$.

It must be remembered that $\sqrt{ab}=\sqrt{a}\sqrt{b}$, *but* $\sqrt{a+b}$ *is not* $\sqrt{a}+\sqrt{b}$.

213. The product of two expressions is found by multiplying each term of one by each term of the other.

Thus $(\sqrt{5}+2\sqrt{6})(3\sqrt{10}-\sqrt{3}) = 3\sqrt{50}-\sqrt{15}+6\sqrt{60}-2\sqrt{18}$

$= 3\sqrt{25}\sqrt{2}-\sqrt{15}+6\sqrt{4}\sqrt{15}-2\sqrt{9}\sqrt{2}$

$= 15\sqrt{2}-\sqrt{15}+12\sqrt{15}-6\sqrt{2} = 9\sqrt{2}+11\sqrt{15}.$

Similar or Like Surds are those which are rational multiples of the same surd; as, for instance, $\frac{1}{2}\sqrt{7}$, $3\sqrt{7}$ and $20\sqrt{7}$ are like surds.

Examples. XXXVI. a.

1. Transform into entire surds $3\sqrt{2}$, $5\sqrt{2}$, $3\sqrt{5}$.
2. ..... $7\sqrt{3}$, $6\sqrt{\frac{1}{2}}$, $10\sqrt{\frac{1}{5}}$.

Simplify

3. $\sqrt{12}$, $\sqrt{8}$, $\sqrt{32}$, $\sqrt{75}$.
4. $\sqrt{245}$, $\sqrt{243}$, $\sqrt{450}$, $\sqrt{175}$.
5. $\sqrt[3]{16}$, $\sqrt[3]{32}$, $\sqrt{500}$, $\sqrt{507}$.
6. $\sqrt{392}$, $\sqrt{2187}$, $\sqrt[3]{2187}$, $\sqrt[3]{-648}$.
7. $\sqrt{1250}$, $\sqrt[3]{81}$, $\sqrt[3]{1029}$, $\sqrt[3]{1024}$, $\sqrt[3]{-250}$.
8. $\sqrt[3]{320}$, $\sqrt[5]{64}$, $\sqrt[3]{375}$, $\sqrt[5]{-486}$, $\sqrt[5]{a^7b^8}$.
9. $\sqrt{a^3b^4c^5}$, $\sqrt[3]{a^4b^7}$, $\sqrt{a^2b-2ab^2+b^3}$.
10. $\sqrt{12}+2\sqrt{48}+5\sqrt{147}-4\sqrt{3}$.
11. $3\sqrt{125}-2\sqrt{80}+\sqrt{378}$.
12. $3\sqrt{24}-5\sqrt{54}+\sqrt{150}$.
13. $\sqrt{80}+2\sqrt{245}-\sqrt{3125}$.
14. $\sqrt[3]{2187}-2\sqrt[3]{24}$.
15. $\sqrt[3]{108}+10\sqrt[3]{32}+\sqrt[3]{500}$.
16. $(5\sqrt{3}-4\sqrt{2})(3\sqrt{3}+2\sqrt{2})$.
17. $(6\sqrt{7}+\sqrt{15})(\sqrt{7}-\sqrt{3})$.
18. $(9\sqrt{2}+5\sqrt{3})(9\sqrt{2}-5\sqrt{3})$.
19. $(3\sqrt{5}+2\sqrt{3})(3\sqrt{5}-2\sqrt{3})$.

214. Results in surds are practically most useful when expressed as decimals, any required degree of accuracy being obtained by the number of places to which they are calculated.

To avoid having a large and unterminated decimal in the denominator of a fraction, we **rationalise the denominator** by multiplying numerator and denominator by a suitable factor. $x\sqrt{y}$ can be rationalised by being multiplied by $\sqrt{y}$.

The factor which will rationalise $\sqrt{a}+\sqrt{b}$ is $\sqrt{a}-\sqrt{b}$: for $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$.

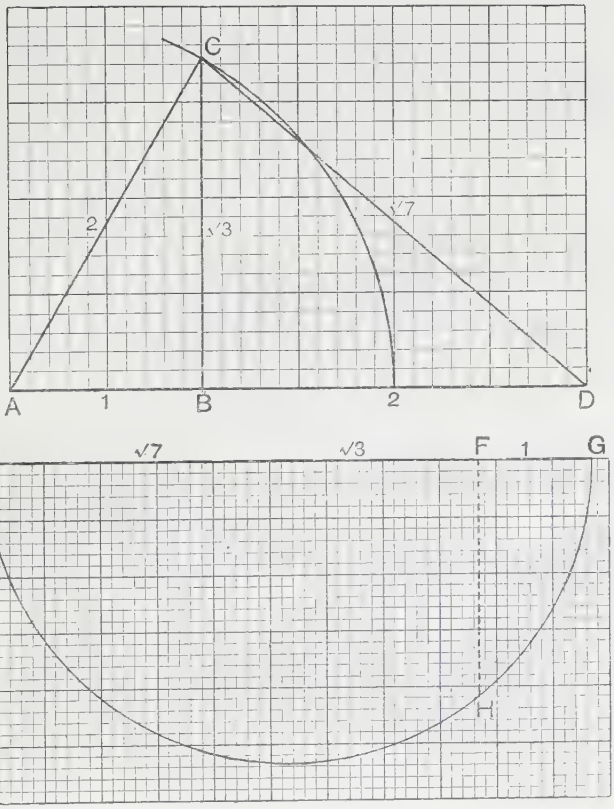
$$\frac{5}{\sqrt{2}} = \frac{5}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2} = \frac{5\times 1{\cdot}414}{2} = \frac{14{\cdot}14}{4} = 3{\cdot}535.$$

$$\frac{6}{\sqrt{5}+\sqrt{3}} = \frac{6}{\sqrt{5}+\sqrt{3}}\cdot\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{6(\sqrt{5}-\sqrt{3})}{5-3} = 3(\sqrt{5}-\sqrt{3}).$$

Division may be performed in the same way:

$$(3+2\sqrt{5})\div(2\sqrt{5}-1)=\frac{3+2\sqrt{5}}{2\sqrt{5}-1}\cdot\frac{2\sqrt{5}+1}{2\sqrt{5}+1}=\frac{8\sqrt{5}+23}{19}.$$

Example. Find geometrically the value of $\sqrt{\sqrt{3}+\sqrt{7}}$.



Take AB of length 1, and BC perpendicular to it. With centre A and radius 2 draw a circle cutting BC at C. Then $CB=\sqrt{3}$. Produce AB to D so that BD = 2. Then $CD=\sqrt{CB^2+BD^2}=\sqrt{7}$.

Draw a line EF of length $\sqrt{3}+\sqrt{7}$ and produce it to G making FG equal to 1. On EG describe a semicircle EHG, and draw FH perpendicular to EG.

$$FH^2=EF\,.\,FG=\sqrt{3}+\sqrt{7}.\quad\therefore\ \sqrt{\sqrt{3}+\sqrt{7}}=FH=2{\cdot}1.$$

Examples. XXXVI. b.

Simplify **1.** $6\sqrt{\frac{1}{3}}$. **2.** $25\sqrt{\frac{1}{12\frac{1}{2}}}$. **3.** $\sqrt{\frac{7}{8}}$.

Express as decimals correct to 2 places

4. $\sqrt{2}+\sqrt{3}$. **5.** $\sqrt{72}-\sqrt{50}$. **6.** $\sqrt{27}-\sqrt{12}$.

7. $\dfrac{6}{\sqrt{2}}$. **8.** $\dfrac{5}{2\sqrt{3}}$. **9.** $\dfrac{28}{\sqrt{24\frac{1}{2}}}$.

10. $1/(\sqrt{7}+\sqrt{3})+1/(\sqrt{7}-\sqrt{3})$. **11.** $(1+\sqrt{2}+\sqrt{3})(1+\sqrt{2}-\sqrt{3})$.

Simplify

12. $(\sqrt{7}+\sqrt{5})(\sqrt{3}-1)(\sqrt{7}-\sqrt{5})(\sqrt{3}+1)$.

13. $(\sqrt{3}+1)(\sqrt{3}-1)(\sqrt{5}+1)(\sqrt{10}-\sqrt{2})$.

14. Find, without extracting roots, which is the greater, $5\sqrt{2}+\sqrt{5}$ or $3\sqrt{7}$.

15. Find also their difference to 2 decimal places by taking sq. roots.

16. Find a factor which will rationalise $(\sqrt{3}+\sqrt{5}+1)(\sqrt{3}+\sqrt{5}-1)$.

17. Divide 1 by $2\sqrt{5}+\sqrt{3}$, and $2\sqrt{15}-3\sqrt{2}$ by $\sqrt{15}-\sqrt{2}$.

18. Square $2\sqrt{3}+\sqrt{5}$, $3\sqrt{7}-\sqrt{2}$, $2\sqrt{5}+4\sqrt{3}$, $\sqrt{3}-\sqrt{3}+\sqrt{2}$.

19. Divide 57 by $5\sqrt{3}-3\sqrt{2}$, and $10+\sqrt{20}$ by $5+\sqrt{5}$.

20. Divide 1 by $\frac{1}{4}(\sqrt{5}-1)$.

Find geometrically *the value (in decimals) of*

21. $\sqrt{2}$. **22.** $\sqrt{3}$. **23.** $\sqrt{7}$, $\sqrt{8}$, $\sqrt{5}$. **24.** $\sqrt{11}$. **25.** $\sqrt{13}$. **26.** $\sqrt{21}$.

27. Draw a length $10+2\sqrt{21}$: make a square equal to a rectangle whose sides are $10+2\sqrt{21}$ and 1; and so find $\sqrt{10+2\sqrt{21}}$. Measure $\sqrt{3}+\sqrt{7}$, and compare the results.

28. In the same way find geometrically $\sqrt{5+2\sqrt{6}}$.

29. On squared paper represent $\sqrt{5}$. Construct a rectangle whose sides are $\sqrt{5}+1$, $\sqrt{5}-1$; and show that the product is 4.

Rationalise the denominators of

30. $\dfrac{1}{3+2\sqrt{2}}$. **31.** $\dfrac{4}{\sqrt{5}+1}$. **32.** $\dfrac{3}{\sqrt{5}+\sqrt{2}}$. **33.** $\dfrac{2+4\sqrt{7}}{2\sqrt{7}+1}$.

34. $\dfrac{5+2\sqrt{6}}{6-2\sqrt{6}}$. **35.** $\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$. **36.** $\dfrac{\sqrt{x+y}+\sqrt{x-y}}{x-\sqrt{x^2-y^2}}$.

Calculate to 2 decimal places the value of

37. $\dfrac{1}{\sqrt{5}-1}$. **38.** $\dfrac{4}{3-2\sqrt{2}}$. **39.** $\dfrac{7\sqrt{2}+\sqrt{3}}{7\sqrt{2}-\sqrt{3}}$. **40.** $\dfrac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}$.

215. If $a+\sqrt{b}=c+\sqrt{d}$, where a, c are rational and $\sqrt{b}$, $\sqrt{d}$ are surds, then $a=c$ and $b=d$.

For if not, let $a=c+x$;

$$\therefore\ x+\sqrt{b}=\sqrt{d};$$

$$\therefore\ \text{by squaring, } x^2+2x\sqrt{b}+b=d;$$

$$\therefore\ 2x\sqrt{b}=d-b-x^2,$$

i.e. a surd = a rational quantity: which is impossible.

$$\therefore\ a=c \text{ and } \sqrt{b}=\sqrt{d}.$$

216. The square of the sum of two surds = a rational quantity + a surd,

$$e.g.\ (\sqrt{5}+\sqrt{3})^2=8+2\sqrt{15}.$$

Consequently the sq. rt. of $a+\sqrt{b}$ may sometimes be found in the form $\sqrt{x}+\sqrt{y}$.

217. If $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ we have, by squaring,

$$a+\sqrt{b}=x+y+2\sqrt{xy}.$$

Equate rational to rational and surd to surd.

Then $a=x+y,\quad \sqrt{b}=2\sqrt{xy}$;

$$\therefore\ a-\sqrt{b}=x+y-2\sqrt{xy}.$$

Thus if $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

218. Find $\sqrt{10+2\sqrt{21}}$.

Let $\sqrt{10+2\sqrt{21}}=\sqrt{x}+\sqrt{y}$.

Then $\sqrt{10-2\sqrt{21}}=\sqrt{x}-\sqrt{y}$.

$\therefore$ by multiplication,

$\sqrt{100-84}=x-y$;

$\therefore\ 4=x-y$.

By squaring we get

$10+2\sqrt{21}=x+y+2\sqrt{xy}$.

By equating rational parts,

$x+y=10$.

But $x-y=4$;

$\therefore\ x=7,\quad y=3$.

$$\therefore\ \sqrt{10+2\sqrt{21}}=\sqrt{7}+\sqrt{3}.$$

Examples. XXXVI. c.

Find the sq. root of

1. $4+2\sqrt{3}$.
2. $7+2\sqrt{6}$.
3. $12-6\sqrt{3}$.
4. $11+6\sqrt{2}$.
5. $30+4\sqrt{14}$.
6. $17-12\sqrt{2}$.
7. $12+2\sqrt{35}$.
8. $32-8\sqrt{15}$.
9. $\frac{9}{4}-\sqrt{5}$.
10. $27+4\sqrt{35}$.
11. $101-28\sqrt{13}$.
12. $\frac{73}{36}+\frac{1}{3}\sqrt{2}$.
13. $4\sqrt{2}-2\sqrt{6}$.
14. $33-18\sqrt{2}$.

Nos. **15–28.** Find the sq. roots of the above expressions to three decimal places by substituting the values of the surds (1) before, (2) after extracting the square roots of the expressions.

Find the 4th roots of

29. $49+20\sqrt{6}$.
30. $17+12\sqrt{2}$.
31. $56-24\sqrt{5}$.
32. $284-48\sqrt{35}$.
33. Find to four places the value of
$$\frac{1}{\sqrt{\{12-\sqrt{(56-24\sqrt{5})}\}}}.$$
34. Find $\sqrt{2}$, $\sqrt{5}$, $\sqrt{7}$ graphically by means of the curve $y=x^2$.
35. Check the results of questions 1, 2, 3 geometrically.

Examples. XXXVI. d.

(*Oral Exercises.*)

1. Simplify (i) $\sqrt{8}$, (ii) $\sqrt{200}$, (iii) $\sqrt[3]{16}$, (iv) $\sqrt{56}$, (v) $\sqrt[3]{250}$, (vi) $\frac{6}{\sqrt{6}}$, (vii) $\frac{14}{\sqrt{7}}$, (viii) $\frac{6}{\sqrt{3}}$, (ix) $\sqrt{8}+\sqrt{18}$, (x) $\sqrt{27}+\sqrt{12}$.
2. Square (i) $2\sqrt{5}$, (ii) $3\sqrt{10}$, (iii) $4\sqrt{3}$, (iv) $2\sqrt{7}$, (v) $5\sqrt{5}$, (vi) $1+\sqrt{3}$, (vii) $\sqrt{3}+\sqrt{2}$, (viii) $\sqrt{5}-1$, (ix) $\sqrt{3}+\sqrt{6}$, (x) $\sqrt{6}-\sqrt{3}$.
3. Give the factor which will rationalise (i) $2\sqrt{3}-\sqrt{7}$, (ii) $5\sqrt{3}+2\sqrt{5}$, (iii) $8-2\sqrt{15}$, (iv) $1+\sqrt{5}$, (v) $20-8\sqrt{2}$, (vi) $12\sqrt{3}-6\sqrt{2}$.

Examples. XXXVI. e.

1. Extract the sq. rt. of $17+4\sqrt{15}$.
2. Simplify $\left(\frac{10+9\sqrt{5}}{9+2\sqrt{5}}\right)^2$.
3. Calculate to 3 places $\frac{(3+\sqrt{5})^2-(2+\sqrt{10})^2}{3-\sqrt{8}}$.
4. Simplify $\frac{\sqrt{3}-1}{\sqrt{5}-2}\times\frac{\sqrt{10}-2\sqrt{2}}{3-\sqrt{3}}$, and find its value to 3 places.
5. Find the value of $\left(\frac{x+a}{x-a}\right)^2-\frac{x}{3a}$ when $x=a(1+2\sqrt{3})$.
6. Reduce $\sqrt{6-\sqrt{17-12\sqrt{2}}}$ to its simplest form.
7. Simplify $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{500}-\sqrt{200}}$ and $\frac{5}{\sqrt{15}+\sqrt{6}}-\frac{1}{\sqrt{60}-\sqrt{24}}$.
Find the value of their product to 3 decimal places.
8. Simplify $\frac{\sqrt{245}+\sqrt{75}}{\sqrt{5}-\sqrt{3}}+\frac{\sqrt{245}-\sqrt{75}}{\sqrt{5}+\sqrt{3}}$.

9. If $a(b-c)^2=c(b+c)^2$, find the value of $\frac{c}{b}\cdot\frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}$.

10. Simplify $\frac{2(1+\sqrt{3})}{1-\sqrt{2+\sqrt{3}}}$, and $\sqrt{1+\sqrt{1-a^2}}+\sqrt{1-\sqrt{1-a^2}}$.

Find the product of

11. $\sqrt{2+\sqrt{3}}$, $\sqrt{2+\sqrt{2+\sqrt{3}}}$, $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$, $\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$, the positive value of the root being taken in each case.

CHAPTER XXXVII.

INDICES.

219. "What is the meaning of a^5?" A very common answer is "a multiplied by itself 5 times." Of course this is wrong. There are 5 factors, but only 4 multiplications. a^5 is the product of 5 factors, each factor being a.

220. $a^n=a\,.\,a\,.\,a\,\ldots\,n$ factors, $a^p=a\,.\,a\,.\,a\,\ldots\,p$ factors.

$\therefore\ a^n\,.\,a^p=a\,.\,a\,.\,a\,\ldots\,n$ factors $\times\,a\,.\,a\,.\,a\,\ldots\,p$ factors

$=a\,.\,a\,.\,a\,\ldots\,\overline{n+p}$ factors $=a^{n+p}$. ..... (1)

$$\frac{a^n}{a^p}=\frac{a\,.\,a\,.\,a\,\ldots\,n \text{ factors}}{a\,.\,a\,.\,a\,\ldots\,p \text{ factors}}.$$

If there are more factors in the numerator than in the denominator, cancelling will leave $n-p$ factors in the numerator: but if there are more in the denominator, cancelling will leave $p-n$ factors in the denominator.

Thus $$\frac{a^n}{a^p}=a^{n-p} \text{ if } n>p \quad \ldots\ldots (2)$$

$$=\frac{1}{a^{p-n}} \text{ if } n<p.$$

221. These results depend upon the definition "a^n is the product of n factors a"; and this definition evidently requires that n should be a positive integer. If we wish to employ fractional or negative indices, it is necessary to give a definition which is applicable to them.

DEF. **Fractional and negative indices are defined as being such that they obey this law:—To multiply powers of a quantity add the indices.**

To find the meaning of $a^{\frac{1}{n}}$.

$a^{\frac{1}{2}}$ when multiplied by itself becomes $a^{\frac{1}{2}+\frac{1}{2}}$, *i.e.* a; $\therefore\ a^{\frac{1}{2}} = \sqrt{a}$.

Similarly $a^{\frac{1}{3}}$ when cubed becomes $a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$, *i.e.* a; $\therefore\ a^{\frac{1}{3}} = \sqrt[3]{a}$.

$a^{\frac{1}{n}}$ when raised to the n^{th} power becomes a; $\therefore\ a^{\frac{1}{n}} = \sqrt[n]{a}$.

We therefore denote $\sqrt{a}$ by $a^{\frac{1}{2}}$, $\sqrt[3]{a}$ by $a^{\frac{1}{3}}$, $\sqrt[n]{a}$ by $a^{\frac{1}{n}}$.

$$a^{\frac{3}{4}} = a^{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} = \sqrt[4]{a} \,.\, \sqrt[4]{a} \,.\, \sqrt[4]{a} = (\sqrt[4]{a})^3.$$

To find the meaning of a^0.

$$a^0 = \frac{a^0 \,.\, a^n}{a^n} = \frac{a^{0+n}}{a^n} = \frac{a^n}{a^n} = 1.$$

To find the meaning of a^{-n}.

$$a^{-n} = \frac{a^{-n} \,.\, a^n}{a^n} = \frac{a^{-n+n}}{a^n} = \frac{a^0}{a^n} = \frac{1}{a^n}.$$

222. (i) $a^{\frac{m}{n}} \,.\, a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}}$ by def.

(ii) $a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}}$.

For $a^{\frac{m}{n}-\frac{p}{q}} \,.\, a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}+\frac{p}{q}}$ by def

$= a^{\frac{m}{n}}$.

223. To prove $(a^m)^n = a^{mn}$ for all values of m and n.

(1) When n is a positive integer, $(a^m)^n = a^m \,.\, a^m \,\ldots\, n$ factors

$= a^{m+m+\ldots n \text{ terms}} = a^{mn}$.

(2) When n is a positive fraction.

Since $a = \sqrt[q]{a^q}$,

$$\therefore\ (a^m)^{\frac{p}{q}} = \sqrt[q]{\{(a^m)^{\frac{p}{q}}\}^q} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}}.$$

(3) $$(a^m)^{-s} = \frac{1}{(a^m)^s} = \frac{1}{a^{ms}} = a^{-ms}.$$

This proves the formula when n is negative.

224. $(ab)^n = a^n b^n$ *for all values of* n.

(1) When n is a positive integer, $(ab)^n = ab \,.\, ab \,\ldots$ (the product containing n factors ab) $= a^n b^n$. .....(1)

(2) When n is a positive fraction $\frac{p}{q}$.

$$\left(a^{\frac{p}{q}}b^{\frac{p}{q}}\right)^q = a^p b^p \text{ by (1)} = (ab)^p = \left\{(ab)^{\frac{p}{q}}\right\}^q;$$

$$\therefore\ (ab)^{\frac{p}{q}} = a^{\frac{p}{q}}b^{\frac{p}{q}}.$$

(3) $$(ab)^{-r} = \frac{1}{(ab)^r} = \frac{1}{a^r b^r} = a^{-r}b^{-r}.$$

This proves it for a negative index.

Example 1. Multiply $\sqrt[3]{a^2} - 2\sqrt{b^{-1}}$ by $a^{\frac{1}{3}} + b^{-\frac{1}{4}}$.

$$\begin{array}{l} a^{\frac{2}{3}} - 2b^{-\frac{1}{2}} \\ a^{\frac{1}{3}} + b^{-\frac{1}{4}} \\ \hline a - 2a^{\frac{1}{3}}b^{-\frac{1}{2}} \\ \quad + a^{\frac{2}{3}}b^{-\frac{1}{4}} - 2b^{-\frac{3}{4}} \\ \hline a - a^{\frac{2}{3}}b^{-\frac{1}{4}} - 2a^{\frac{1}{3}}b^{-\frac{1}{2}} - 2b^{-\frac{3}{4}} \end{array}$$

Example 2. Multiply $x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{2}}$.

$$\begin{aligned} &\left\{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) + x^{\frac{1}{4}}y^{\frac{1}{4}}\right\} \cdot \left\{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right) - x^{\frac{1}{4}}y^{\frac{1}{4}}\right\} \\ &= \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^2 - \left(x^{\frac{1}{4}}y^{\frac{1}{4}}\right)^2 = x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y - x^{\frac{1}{2}}y^{\frac{1}{2}} \\ &= x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y. \end{aligned}$$

Example 3. Divide $a^2 + 4a^{\frac{3}{2}}b^{-1} + 6ab^{-2} + 4a^{\frac{1}{2}}b^{-3} + b^{-4}$ by $a^{\frac{1}{2}} + b^{-1}$.

$$\begin{array}{l} a^{\frac{1}{2}} + b^{-1}\Big) a^2 + 4a^{\frac{3}{2}}b^{-1} + 6ab^{-2} + 4a^{\frac{1}{2}}b^{-3} + b^{-4}\Big(a^{\frac{3}{2}} + 3ab^{-1} + 3a^{\frac{1}{2}}b^{-2} + b^{-3} \\ \quad a^2 + a^{\frac{3}{2}}b^{-1} \\ \quad \hline \quad 3a^{\frac{3}{2}}b^{-1} + 6ab^{-2} \\ \quad 3a^{\frac{3}{2}}b^{-1} + 3ab^{-2} \\ \quad \hline \qquad 3ab^{-2} + 4a^{\frac{1}{2}}b^{-3} \\ \qquad 3ab^{-2} + 3a^{\frac{1}{2}}b^{-3} \\ \qquad \hline \qquad\quad a^{\frac{1}{2}}b^{-3} + b^{-4} \\ \qquad\quad a^{\frac{1}{2}}b^{-3} + b^{-4} \\ \qquad\quad \hline \end{array}$$

Examples. XXXVII. a.

Express as simply as possible with indices:

1. $\frac{1}{\sqrt[3]{a^{-1}}}$.
2. $\sqrt[5]{a^2b}$.
3. $\frac{3\sqrt{b}}{\sqrt[3]{c}}$.
4. $\frac{a^4}{\sqrt[3]{a^2}}$; $\frac{\sqrt[4]{y}}{\sqrt{xy}}$; $\sqrt{2a^{-3}}$.
5. Simplify $8^{\frac{2}{3}}$ and $25^{-\frac{1}{2}}$.
6. Simplify $27^{-\frac{2}{3}}$ and $49^{\frac{3}{2}}$.
7. Express with positive indices $a^{-1}bc+a^{-2}b^{-1}c+ab^{-1}c^{-1}$.

Simplify

8. $16^{\frac{3}{4}}$.
9. $256^{-\frac{3}{8}}$.
10. $289^{-\frac{1}{2}}$.
11. $32^{-\frac{3}{5}}$.
12. $729^{-\frac{2}{3}}$.
13. $625^{\frac{3}{4}}$.
14. $\frac{1024^{\frac{3}{10}}}{125^{-\frac{2}{3}}}$.
15. $\frac{1}{343^{-\frac{2}{3}}}$.

Multiply

16. $2y+3x^{\frac{1}{4}}y^{\frac{1}{2}}-x^{\frac{1}{2}}$ by $7x^{\frac{1}{4}}+5y^{\frac{1}{2}}$.
17. $\sqrt{x^3}+1+\frac{1}{\sqrt{x^3}}$ by $\sqrt{x^3}-1+\frac{1}{\sqrt{x^3}}$.
18. $a^{-1}+a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$ by $a^{-1}-a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$.
19. $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{3}}$.
20. $a+2b^{\frac{1}{2}}+3c^{\frac{1}{2}}$ by $a+2b^{\frac{1}{2}}-3c^{\frac{1}{2}}$.
21. $x^3+x^{\frac{3}{2}}y+x^{\frac{1}{4}}y^2+y^3$ by $x^{\frac{1}{4}}-y$.
22. $a^2+3ab^{-1}+4c^{-2}$ by $a^2-3ab^{-1}-4c^{-2}$.

Divide

23. $8x^3-27y^{-3}$ by $2x-3y^{-1}$.
24. $x^{-3}-64y^2$ by $x^{-\frac{1}{2}}+2y^{\frac{1}{3}}$.
25. $a-2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$ by $a^{\frac{1}{2}}-2a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.

Find sq. root of

26. $\frac{\sqrt{a}}{\sqrt{b}}+\frac{2\sqrt[4]{a}}{\sqrt[4]{b}}+3+\frac{2\sqrt[4]{b}}{\sqrt[4]{a}}+\frac{\sqrt{b}}{\sqrt{a}}$.
27. $9x^{\frac{2}{3}}+25y+4z^2-\frac{12z}{\sqrt[3]{x^{-1}}}-\frac{20y^{\frac{1}{2}}}{z^{-1}}+30x^{\frac{1}{3}}y^{\frac{1}{2}}$.
28. Prove $(x+x^{-1})^2-(y+y^{-1})^2=(xy-x^{-1}y^{-1})(xy^{-1}-x^{-1}y)$.
29. Given $e=2{\cdot}7183$, $e^{\frac{3}{2}}=4{\cdot}4816$, $e^{-\frac{1}{2}}=\cdot6065$, $e^{-1}=\cdot3679$, $e^{-2}=\cdot1353$, trace the graph of $y=e^x$, and hence find $e^{\frac{1}{2}}$.
30. Tabulate the values of $7^{\frac{1}{2}}$, $7^{\frac{1}{4}}$, $7^{\frac{1}{6}}$, $7^{\frac{1}{8}}$, $7^{\frac{1}{12}}$ to 8, 7, 6, 5, 4 decimal places respectively. If $1{\cdot}063^x=1{\cdot}627$ find x.
Find $7^{\frac{3}{4}}$ and $7^{\frac{5}{16}}$ to 3 places.
31. Given $7^{\frac{1}{32}}=1{\cdot}0627$, find $7^{\frac{1}{128}}$.
32. Prove $2\times7^{\frac{25}{128}}=$ nearly 10^2. Find the difference between these.

Example. Simplify $\dfrac{2^n \cdot 8^{n+2} \cdot 4^{-3n}}{2^{-2n}}$.

Express it in powers of 2.

$$\begin{aligned}\text{The fraction} &= 2^n \cdot (2^3)^{n+2} \cdot (2^2)^{-3n} \cdot 2^{2n} \\ &= 2^n \cdot 2^{3n+6} \cdot 2^{-6n} \cdot 2^{2n} \\ &= 2^{n+3n+6-6n+2n} = 2^6.\end{aligned}$$

Examples. XXXVII. b.

Simplify

1. $(a^{\frac{1}{3}}x^{-2})^{-3}$.
2. $\left(\frac{1}{16}a^{12}b^{-8}\right)^{\frac{1}{4}}$.
3. $(27b^{-6}c^3)^{\frac{1}{3}}$.
4. $(2x^{-1}\sqrt[3]{y^2})^{-6}$.
5. $\left(\dfrac{32a^{-5}}{b^{-10}}\right)^{\frac{1}{5}}$.
6. $\dfrac{a^{-1}+b^{-1}}{a^{-2}-b^{-2}}$.
7. $\dfrac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$.
8. $(1024^{-\frac{4}{5}})^{\frac{1}{2}}$.
9. $\dfrac{2^n \cdot 4^{n+1}}{8^{n-2}}$.
10. $\dfrac{5^{-n} \cdot 25^{2n-2}}{5^{3n-2} \cdot 10^{-1}}$.

Examples. XXXVII. c.

1. Solve $x^{\frac{11}{3}} - 31x^2 = 32\sqrt[3]{x}$.
2. Simplify $\left\{\sqrt[3]{4} \times \dfrac{1}{\sqrt[6]{8}} \times \sqrt[12]{2^{-1}}\right\}^4$.
3. Divide $a^{-\frac{6}{5}} - \frac{1}{4}a^{-\frac{4}{5}}b^{\frac{1}{2}} + 3a^{-\frac{2}{5}}b^{\frac{1}{4}} - \frac{3}{4}b^{\frac{3}{4}}$ by $a^{-\frac{4}{5}} + 3b^{\frac{1}{4}}$, and express the answer with radical signs.
4. Find the square root of $\sqrt[5]{b^8} - 2b^{\frac{11}{5}}c^{-\frac{3}{5}} + 2\sqrt[5]{(bc)^4} + b^{\frac{14}{5}}c^{-\frac{6}{5}} - 2b^{\frac{7}{5}}c^{\frac{1}{5}} + \sqrt[5]{(bc)^2}$.
5. Divide $a^3 + b^{-3}$ by $a^{\frac{1}{2}} - b^{-\frac{1}{2}}$.
6. Give the product of $a^{\frac{1}{3}} + a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}$ and $a^{\frac{1}{3}} - a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}$.
7. Factorise $\frac{5}{4}(a^{\frac{3}{2}} + a^{-\frac{3}{2}}) + \frac{1}{4}(a^{\frac{1}{2}} + a^{-\frac{1}{2}})$.
8. Square $a^{\frac{1}{2}} - 2a^{\frac{1}{4}}b^{\frac{1}{2}} + b$, and divide the result by $(a^{\frac{1}{4}} - b^{\frac{1}{2}})^3$.
9. Simplify $\dfrac{x^{\frac{4}{3}} - 4y^{-\frac{4}{3}}}{x^{\frac{1}{3}} + 4x^{\frac{2}{3}}y^{-\frac{2}{3}} + 4y^{-\frac{4}{3}}}$.
10. Divide $x^{12} - x^{-12} + 6(x^8 - x^{-8}) + 9(x^4 - x^{-4})$ by $x^6 - x^{-6} + 3(x^2 - x^{-2})$ by factorising the dividend.
11. Prove that $7^{\frac{1}{4}} - 7^{\frac{1}{8}} = \cdot 35$ nearly.

CHAPTER XXXVIII.

RATIO, PROPORTION AND VARIATION.

225. If two quantities are of the same kind, they have a **ratio**; and the ratio of the 1st to the 2nd is the quotient obtained by dividing the 1st by the 2nd, whether that quotient be integral or fractional.

The ratio of a to b is expressed as $a : b$ or $\frac{a}{b}$.

a, b are respectively called the 1st and 2nd *terms* or *members* of the ratio, or the **antecedent** and **consequent.**

If the antecedent = the consequent, the ratio is a **ratio of equality,** and is equal to unity.

If the antecedent is the greater, the ratio is called a **ratio of greater inequality,** *i.e.* an improper fraction.

If the antecedent is the less, the ratio is a **ratio of less inequality,** *i.e.* a proper fraction.

226. *A ratio of greater inequality is diminished by adding the same positive quantity to both its members.*

Let $\frac{a}{b}$ be a ratio of greater inequality, $\frac{a+c}{b+c}$ the new ratio.

These can be compared by being multiplied by their common denominator. Thus

$$\frac{a}{b} \times b(b+c) = ab + ac, \quad \text{and} \quad \frac{a+c}{b+c} \times b(b+c) = ab + bc,$$

But by hypothesis $ac > bc$; $\therefore \frac{a}{b} > \frac{a+c}{b+c}$,

i.e. the ratio has been diminished by the addition of c to both its members.

Similarly it may be proved that *a ratio of less inequality is increased by adding the same positive quantity to both its members.*

The proof of this should be written out as an exercise.

The two statements may be combined in one, viz.: *A ratio is made nearer to unity by adding the same positive quantity to both its members.*

227. Ratios are *compounded* by being multiplied together.

The **duplicate** ratio of a to b is $a^2 : b^2$.

The **sub-duplicate** ratio of a to b is $a^{\frac{1}{2}} : b^{\frac{1}{2}}$.

228. Many properties of ratios are easily proved by taking some single letter k to represent a ratio, or to represent each of several equal ratios.

Example. To prove that, if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, each of these ratios $= \frac{la+mc+ne}{lb+md+nf}$.

Let $\frac{a}{b} = k$. Then $\frac{c}{d} = k$, and $\frac{e}{f} = k$;

$\therefore\ a = bk, \quad c = dk, \quad e = fk.$

$$\frac{la+mc+ne}{lb+md+nf} = \frac{lbk+mdk+nfk}{lb+md+nf} = \frac{(lb+md+nf)k}{lb+md+nf} = k = \frac{a}{b}.$$

From this it is evident that **if a number of ratios are equal, a new ratio equal to each of them can be formed by adding their antecedents for a new antecedent and adding their consequents for a new consequent.**

229. If $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ are unequal, the ratio $\frac{a+c+e}{b+d+f}$ lies in magnitude between the greatest and least of these ratios.

Suppose $\frac{a}{b} > \frac{c}{d} > \frac{e}{f}$.

Let $\frac{a}{b} = k$. Then $\frac{c}{d} < k$ and $\frac{e}{f} < k$;

$\therefore\ a = bk, \quad c < dk, \quad e < fk;$

$\therefore\ a+c+e < (b+d+f)k; \quad \therefore\ \frac{a+c+e}{b+d+f} < k.$

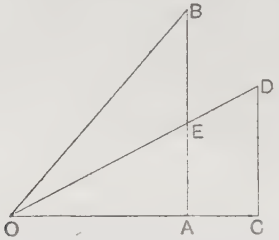
Let $\frac{e}{f} = k'$. Then in a similar way $\frac{a+c+e}{b+d+f} > k'$.

Thus $\frac{a+c+e}{b+d+f}$ lies between the greatest and least of the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$.

Graphic Representation of Ratio.

230. Take an abscissa OA to represent the consequent of the ratio, and an ordinate AB to represent the antecedent on the

same scale. The magnitude of the angle AOB enables us to estimate whether the ratio is greater or less than another ratio represented in the same manner.



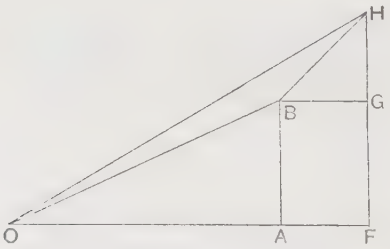
Suppose $\frac{DC}{OC}$ to be the 2nd ratio; and let OD meet AB at E.

By similar $\triangle^s$ EAO, DCO $\quad \frac{DC}{OC} = \frac{EA}{OA}$.

$\therefore$ $\frac{EA}{OA}$ also represents the 2nd ratio.

Thus we can compare the ratios by means of the lengths of AB and AE.

A ratio of less inequality is increased by adding the same quantity to both its terms.



As before take an abscissa OA and ordinate AB, so that $\frac{AB}{OA}$ represents a ratio.

If OA be produced to F, and an ordinate FGH be drawn, where FG = AB and GH = AF, the ratio $\frac{AB}{OA}$ has been altered by adding the same quantity HG or AF to both its terms.

The new ratio $\frac{\mathsf{HF}}{\mathsf{OF}}$ > the old ratio, if OH cuts AB above B; *i.e.* if the $\angle \mathsf{HBG} >$ the $\angle \mathsf{BOA}$.

But $\angle \mathsf{HBG} = 45^\circ$;

$\therefore$ the new ratio > the old, if $\angle \mathsf{BOA} < 45^\circ$; *i.e.* if the ratio $\frac{\mathsf{AB}}{\mathsf{OA}}$ is one of less inequality.

Note. It is convenient to have a name to denote the graphic representation of a ratio. In trigonometry the ratio AB : OA is called the *tangent* of the angle AOB. In many mathematical works this ratio is denoted by the *slope* of the line OB.

231. If $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ are unequal ratios, the ratio $\frac{a+c+e}{b+d+f}$ lies in value between the greatest and least of them.

Draw abscissae OB, OD, OF to represent b, d, f, and ordinates BA, DC, FE to represent a, c, e.

Draw an abscissa OQ to represent $b+d+f$, and an ordinate QP to represent $a+c+e$. It will be found that the angle POQ lies in value between the greatest and least of the angles AOB, COD, EOF.

$\therefore$ the ratio $\frac{a+c+e}{b+d+f}$ lies in value between the greatest and least of the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$.

Example 1. If $\frac{7x-3y}{5x+4y}=\frac{29}{33}$, find the ratio $x : y$.

$$33(7x-3y)=29(5x+4y);$$

$$\therefore\ 231x-99y=145x+116y;$$

$$\therefore\ 86x=215y;$$

$$\therefore\ x=\frac{215y}{86};$$

$$\therefore\ \frac{x}{y}=\frac{215}{86}=\frac{5}{2}.$$

Example 2. If $2x^2-5xy+2y^2=0$, find the ratio $x : y$.

$$2\left(\frac{x}{y}\right)^2-5\frac{x}{y}+2=0;$$

$\therefore$ by solving this quadratic equation we get

$$\frac{x}{y}=2 \text{ or } \frac{1}{2}.$$

Example 3. When a straight line is divided in extreme and mean ratio, what are approximately the ratios of the parts to the whole?

Let the whole measure a units, the two parts x and $a-x$ units.

By hypothesis $x^2=a(a-x)$;

$\therefore\ x^2+ax=a^2$;

$\therefore\ x=a.\dfrac{\sqrt{5}-1}{2}$, rejecting the negative solution.

$\therefore$ the ratio $\dfrac{x}{a}=\dfrac{\sqrt{5}-1}{2}=\cdot 618$ approximately.

For a very rough approximation we may call the two parts three-fifths and two-fifths of the whole.

Example 4. Find a ratio intermediate between $\frac{7}{8}$ and $\frac{15}{18}$.

By what has been proved we see that such a ratio can be found by adding the numerators and adding the denominators. Result $\frac{22}{24}$, *i.e.* $\frac{11}{12}$.

NOTE.—A ratio may sometimes be simplified by the use of Art. 229 for purposes of approximation or checking.

Thus $\dfrac{4526}{1007}=$(approximately) $\dfrac{4494\cdot 5}{1000}=4\cdot 4945.$

The fuller working is

$$\frac{4526}{1007}=\text{roughly }\frac{31\cdot 5}{7}=\frac{4526-31\cdot 5}{1007-7}=\frac{4494\cdot 5}{1000}=4\cdot 4945.$$

Examples. XXXVIII. a.

1. By means of squared paper compare the ratio $\frac{11}{19}$ with $\frac{17}{30}$, $\frac{7}{8}$, $\frac{14}{23}$, $\frac{7}{11}$, $\frac{33}{57}$.

[Join the points (0, 0), (19, 11) and produce this line. Observe whether the point (30, 17), for example, is above or below this line.]

2. In the same way compare a ratio with what it becomes when its antecedent and consequent are increased by the same quantity.

3. Examine the result when antecedent and consequent are diminished by the same quantity.

4. Find the ratio of 2 lbs. 6 oz. to 3 lbs. 9 oz.

5. What value of x will make $\dfrac{27+x}{37+x}$ equal $\dfrac{2}{3}$?

6. If $\dfrac{5x-4y}{3x-2y}=4$, find the ratio of x to y.

7. If $\dfrac{x}{5}=\dfrac{y}{8}$, find the ratio of $x+5$ to $y+8$.

8. $\dfrac{x}{y}=\dfrac{3}{5}$. Find the value of $\dfrac{x+y}{y-x}$.

9. $3x^2-10xy+3y^2=0$. Find $\dfrac{x}{y}$.

10. $\frac{x+6}{x+15}$ = duplicate ratio of 4 to 5. Find x.

11. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove $\frac{3a^3+4a^2c+5c^2e}{3b^3+4b^2d+5d^2f}=\frac{a^3}{b^3}$.

12. $\frac{a}{x+y}=\frac{b}{y-z}=\frac{c}{z+x}$. Prove $a=b+c$.

13. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove $\frac{a^4+5c^3e+e^4}{b^4+5d^3f+f^4}=\frac{a^2c^2}{b^2d^2}$.

14. What quantity must be added to each term of the ratio $a:b$ to make it equal to $c:d$?

15. If $\frac{x-y}{y+z}=\frac{y-z}{x+y}=\frac{x+z}{x-z}$, each of these ratios $=\frac{x}{x+y}$.

16. Find a ratio intermediate between $\frac{2}{2}\frac{1}{9}$ and $\frac{6}{8}\frac{5}{9}$.

17. If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, prove that each of these ratios $=\frac{ab+bc+cd}{b^2+c^2+d^2}$, and that $\frac{a}{c}=\frac{a^2+b^2+c^2}{b^2+c^2+d^2}$.

18. If an object of height h at a distance d from the observer subtends a *small* angle of A degrees at his position, it may be proved that roughly $h=Ad/57{\cdot}3$. Use this to find the height of a tower which subtends an angle of 9° at a point 170 yds. away.

19. If $\frac{(p-1)ab}{pb-a}$ be taken away from each member of the ratio $\frac{a}{b}$, the new ratio is $\frac{a}{pb}$.

Examples. XXXVIII. b.

Graphic Examples of Ratio.

1. On squared paper represent the ratios $\frac{3}{5}$, $\frac{5}{9}$, $\frac{6}{11}$, and see which is greatest and which least.

2. By observing where the hypotenuse of each cuts the ordinate whose abscissa is 10 find the value of each ratio in a decimal form.

3. Draw the ratio formed by the sum of antecedents ÷ the sum of consequents.

4. Draw the ratio formed by adding 6 to the numerator and denominator of $\frac{1}{1}\frac{5}{6}$. Compare it with $\frac{1}{1}\frac{5}{6}$.

5. Two men share £20 in the ratio 5 : 3. Find their shares graphically.

6. Measure abscissae OA, OC equal to 6 and 11. Measure ordinates AB, CD equal to 13 and 18. Find what angle the line BD makes with the line OC.

7. Find graphically what number must be added to each member of the ratio $\frac{2}{3}\frac{5}{3}$ to make it $\frac{5}{6}$.

[The previous question suggests the method.]

8. By squared paper reduce $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$ to ratios whose denominator is 24.

9. What number must be taken from each member of the ratio $\frac{4}{5}\frac{7}{9}$ to make it $\frac{1}{1}\frac{5}{1}$?

10. The string of a kite makes an angle of 30° with the horizontal ground. Find by measurement the ratio of the height of the kite to the length of string.

11. **Show graphically** that, if $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ are all equal, the ratio $\frac{a+c+e}{b+d+f}$ is equal to any one of them, by taking a, c, e for ordinates and b, d, f for abscissae.

12. **Show graphically** that, if $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ are unequal, $\frac{a+c+e}{b+d+f}$ lies in value between the greatest and least of them.

Examples. XXXVIII. c.

1. Divide a given length in the ratio of 3 to 2 geometrically.

2. Construct a scale of feet for a drawing in which 10 ft. 6 in. is represented by $3\frac{1}{2}$ inches.

3. What ratio does the area of the drawing bear to the area of the figure represented?

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove the following (4 to 8):

4. $\frac{3a^2-5ce+7e^2}{3b^2-5df+7f^2}=\frac{ace}{bdf}$. 5. $\frac{ac+2c^2}{bf+2d^2}=\frac{a^2+c^2}{b^2+d^2}$.

6. $\frac{la+mc+ne}{lb+md+nf}=\frac{la-mc+ne}{lb-md+nf}=\frac{a}{b}$. 7. $\frac{a^3+c^3}{b^3+d^3}=\frac{ace}{bdf}$.

8. $\sqrt[3]{\frac{pa^2c+qade^2+re^3}{pb^2d+qbdf^2+rf^3}}=\frac{a}{b}$.

9. If $\frac{a}{b}=\frac{3}{5}$, find the value of $\frac{7a+2b}{4a+10b}$.

10. If $\frac{a}{b}=\frac{3}{4}$, and $\frac{c}{d}=\frac{5}{8}$, find the value of $\frac{3ac+5bd}{4ac+8bd}$.

11. If $\frac{2x+6y}{5x+7y}=\frac{4}{7}$, find $\frac{x}{y}$.

12. If $\frac{a+b}{a-b}=\frac{c+d}{c-d}$, prove $ad=bc$.

13. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a+b}{b}=\frac{c+d}{d}$ and $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

14. If $\frac{a}{b}=\frac{b}{c}$, prove $a^3+c^3=\left(a+\frac{b^2}{a}\right)(a^2-b^2+c^2)$.

15. Find two numbers whose sum is 85 and whose ratio is 8 : 9.

16. Divide 92 graphically so that the two parts may be in the ratio 8 : 15.

17. Divide 65 graphically into two parts so that $\frac{3}{5}$ of one may be $\frac{5}{9}$ of the other.

18. If $\frac{x+2y}{3x-z}=\frac{2x-z}{3y+x}=\frac{5y-3x+z}{4y-4x+z}$, prove each of these ratios = 1.

19. If $\frac{x}{a}=\frac{y}{b+c}=\frac{z}{b+c-a}$, prove $x-y+z=0$.

20. If $\frac{a}{x-y}=\frac{b}{y-z}=\frac{c}{z-x}$, prove $a+b+c=0$.

21. If $\frac{a}{b}=\frac{b}{c}$, prove $\frac{a}{c}=\frac{a^2}{b^2}=\frac{a^2+b^2}{b^2+c^2}$.

22. If $\frac{mx+ny}{ma+nb}=\frac{px+qy}{pa+qb}$, each fraction $=\frac{x}{a}=\frac{y}{b}$.

23. At present A's age is to B's age as 5 to 2, but in 30 years' time the ratio will be 35 : 23. Find their ages.

24. If $7(x-y)=3(x+y)$, what is the ratio of x to y?

25. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $a+3c+2e : a-e=b+3d+2f : b-f$.

26. If $8(x+y)=11(x-y)$, what is the ratio of x to y?

27. On a certain map a road 1320 yds. long is represented by $2\frac{3}{4}$ inches. Determine the scale of the map. What area on the map would represent $\frac{1}{10}$ sq. mile?

28. Find the ratio of x to y from the equation $2x^2-9xy+10y^2=0$.

29. Two vessels contain mixtures of wine and water in the ratios of 8 to 3 and 5 to 1 respectively. In what ratio must liquid be drawn from each vessel to give a mixture in the ratio 4 to 1?

30. In a certain examination the number of those who passed was 3 times the number of those who failed. If there had been 16 fewer candidates and if 6 more had failed, the numbers would have been as 2 to 1. Find the number of candidates.

31. If $\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{z+y}{x+z}$, each of these ratios $=\frac{x}{y}$, unless $b+c=0$.

32. Two men set out at the same time from A and B along a road ABC, both going in the direction BC. The hinder man travels at $\frac{4}{5}$ of the pace of the other and overtakes him at a point 10 miles from B. Find the distance AB.

33. Solve this question also graphically.

34. The marks gained in an examination-paper for which the maximum was 65 were 53, 42, 37. Find by a diagram what these would be if the maximum were 100.

PROPORTION.

232. The equality of two ratios forms a **proportion**. Thus a, b, c, d are in proportion if $\frac{a}{b}=\frac{c}{d}$. Quantities are in **continued proportion**, if the ratio of the 1st to the 2nd = the ratio of the 2nd to the 3rd = the ratio of the 3rd to the 4th, and so on:

$$e.g.\ \frac{1}{2}=\frac{2}{4}=\frac{4}{8}=\dots.$$

If $\frac{a}{b}=\frac{c}{d}$, $ad=bc$; that is, *in any proportion the product of the extremes = the product of the means.*

If $\frac{a}{b}=\frac{b}{c}$, $ac=b^2$; *i.e. if three quantities are in continued proportion, the product of the 1st and 3rd = the square on the 2nd.*

In this case b is said to be a **mean proportional** between a and c, and c a **third proportional** to a and b.

If $\frac{a}{b}=\frac{b}{c}$, then $\frac{a}{c}=\frac{a}{b}\cdot\frac{b}{c}=\frac{a^2}{b^2}$; *i.e. if three magnitudes are in continued proportion the ratio of the 1st to the 3rd is the duplicate ratio of the 1st to the 2nd.* (See Art. 227.)

If $\frac{a}{b}=\frac{c}{d}$, the following results are important and easily deducible:

(1) $\frac{a}{c}=\frac{b}{d}$. **(Alternando.)** (2) $\frac{b}{a}=\frac{d}{c}$. **(Invertendo.)**

(3) $\frac{a}{b}+1=\frac{c}{d}+1$, or $\frac{a+b}{b}=\frac{c+d}{d}$. **(Componendo.)**

(4) $\frac{a}{b}-1=\frac{c}{d}-1$, or $\frac{a-b}{b}=\frac{c-d}{d}$. **(Dividendo.)**

(5) $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ **(Componendo and Dividendo.)**

A large number of questions in proportion may be solved by the 'k' method explained in Art. 228.

Geometrical Methods.

233. *Find a mean proportional to two given numbers.*

The two numbers can be represented by straight lines, and a mean proportional can be found as in *Geometry*, v. 24.

234. *Find a fourth proportional to three given numbers.*

Represent them by straight lines and proceed as in *Geometry*, v. 23.

Examples. XXXVIII. d.

1. If $ab=cd$, express this in the form of a proportion.

2. Find a mean proportional to 7 and 63; and a 3rd proportional to $2x$ and $5x^3$.

3. Find a 4th proportional to a, ab, c; to $3c$, $4y$, $9cx$; and to a^3, ab^2c, bc^2.

If $\frac{a}{b}=\frac{c}{d}$, prove the relations 4–7.

4. $\frac{3a^2+c^2}{3b^2+d^2}=\frac{5a^2-2c^2}{5b^2-2d^2}$.

5. $\frac{a^2+ab+b^2}{a^2-ab+b^2}=\frac{c^2+cd+d^2}{c^2-cd+d^2}$.

6. $\frac{la^2+mc^2}{lb^2+md^2}=\frac{ac}{bd}$.

7. $\sqrt[5]{\frac{2ac^4+3c^5}{2bd^4+3d^5}}=\frac{a}{b}$.

8. If $a:b=b:c$, $a^3+b^3=a(a+b)(a-b+c)$, and
$a^4+a^2c^2+c^4=(a^2+b^2+c^2)(a^2-b^2+c^2)$.

9. Find two numbers such that their sum, product, and difference of squares are proportional to 7, 12, 7.

10. Three numbers are in continued proportion; the middle one is 15 and the sum of the others 50. Find them.

11. Find a third proportional to $\sqrt{3}+1$ and $\sqrt{3}+2$.

12. Find a mean proportional between $\sqrt{5}+\sqrt{2}$ and $\frac{12}{\sqrt{5}-\sqrt{2}}$.

13. If $(a+b+c+d)(a-b-c+d)=(a+b-c-d)(a-b+c-d)$, then a, b, c, d are in proportion.

If $a:b=b:c$, prove the following (14, 15, 16):

14. $a-b:b-c=b:c$.

15. $a:c=a^2+b^2:b^2+c^2$.

16. $(a+b+c)(b-c)=ab-c^2$.

17. If $a:b=b:c$, and if $b-c=\frac{2b}{a}$, prove that $a-c=\frac{2(a+b)}{a}$.

18. If $a^2-b^2=(x+y)^2$, put this in the form of a proportion.

19. If a, x, $a-x$ are in continued proportion, find x. Give also a geometrical construction.

20. If $a-b:b-c=b:c$, then a, b, c are in continued proportion.

21. Find *geometrically* a 4th proportional to 1·4, 2·1, 1·8.

..... mean proportional to 12 and 6.

22. The components of gunpowder are:—nitre 75 per cent., charcoal 15 per cent., and sulphur 10 per cent. How many grams of each (to the nearest gram) are needed to make a pound (454 grams) of gunpowder?

23. Find 2 numbers such that their sum, their difference, and the sum of their squares are proportional to 5, 3, 51.

24. Given $a+b:a-b=c+d:c-d$ express the ratio
$$\sqrt{a^2+pab+b^2}:\sqrt{c^2+pcd+d^2}$$
in terms of a and c alone.

25. If $x+2y : a+3b = y+3x : a-4b$, prove that $x : y = a+5b : 2a+5b$, and that $y+2x : x+3y = 4a+15b : 7a+20b$.

26. If $p : q = a^2 : b^2$, and $a : b = (a+p)^{\frac{1}{2}} : (a-q)^{\frac{1}{2}}$, find in terms of a and p the value of $\dfrac{p-q}{q}$.

27. If $b(a-c) : c(b-d) = a-b : c-d$, then either $b=c$ or $ad=bc$.

28. If $a : b = c : d$, then $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

29. If a, b, c, d are in continued proportion

$$a^3+b^3 : b^3+c^3 = b^3+c^3 : c^3+d^3.$$

30. If $\dfrac{a}{b}=\dfrac{c}{d}$, $\left(\dfrac{1}{a}+\dfrac{1}{d}\right)-\left(\dfrac{1}{b}+\dfrac{1}{c}\right)=\dfrac{(a-b)(a-c)}{abc}$.

31. If a, b, c are in continued proportion

$$\frac{a+b+c}{a-b+c}=\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

32. If $a^2+c^2 : ab+cd = ab+cd : b^2+d^2$, prove that $a : b = c : d$.

33. If $a : b = c : d$, $\dfrac{1}{ma}+\dfrac{1}{nb}+\dfrac{1}{pc}+\dfrac{1}{qd}=\dfrac{1}{bc}\left(\dfrac{a}{q}+\dfrac{b}{p}+\dfrac{c}{n}+\dfrac{d}{m}\right)$.

VARIATION.

235. When it is said that a varies as b (written $a \propto b$), it is meant that, however a and b may alter their values, the ratio $a : b$ remains unchanged.

$$\therefore \text{ if } a \propto b, \ \frac{a}{b} = \text{a constant} = m \text{ suppose.}$$

Thus if $a \propto b$, $a = mb$.

E.g. in a circle the circumference $\propto$ the diameter;

$\therefore$ circumference $\div$ diameter = a constant.

This constant is 3·14159... and is denoted by π.

$\therefore$ circumference $= 2\pi r$.

If $a = \dfrac{m}{b}$ where m is a constant, a is said to vary *inversely* as b.

If $a = mbc$, a varies as b and c jointly.

If $a = \dfrac{mb}{c}$, a varies directly as b, and inversely as c.

The law of gravitation furnishes an example: for the attraction of the earth on an external object varies inversely as the square of the object's distance from the centre of the earth.

236. *If $a \propto b$ when c is constant, and $a \propto c$ when b is constant, then $a \propto bc$ when both b and c are variable.*

[The meaning of this is most easily understood from an example. In a triangle the area $\propto$ the base when the altitude is constant, and varies as the altitude when the base is constant. When both base and altitude are variable, the area $\propto$ base $\times$ altitude.]

Proof. $a \propto b$ when c is constant;

$\therefore\ a = mb$ (where m is independent of b). .....(1)

But $a \propto c$ when b is constant;

$\therefore\ mb \propto c$ when b is constant;

$\therefore\ m \propto c$;

$\therefore\ m = nc$ (where n is independent of c). .....(2)

Also n, being a factor of m, is independent of b.

From (2) substitute in (1).

$\therefore\ a = ncb$ (where n is independent of b and c).

$\therefore\ a \propto bc$.

237. When $y \propto x$, $y = mx$.

The graph of this is a straight line through the origin.

When $y \propto x^2$, $y = mx^2$.

The graph of this is a parabola. (Art. 133.)

When $y \propto \dfrac{1}{x}$, $xy = m$.

The graph of this is a hyperbola. (Art. 166.)

Example. If $a \propto b$, and $a = 10$ when $b = 3$, find b when $a = 1\frac{5}{2}$.

Here $a = mb$, m being a constant. ..... (1)

The statement, that $a = 10$ when $b = 3$, enables us to find m.

For from (1) $\quad 10 = m \times 3$; $\quad \therefore\ m = 1\frac{0}{3}$;

$\therefore$ the relation between a and b is $a = 1\frac{0}{3}b$;

$\therefore$ when $a = 1\frac{5}{2}$ we have $1\frac{5}{2} = \dfrac{10b}{3}$;

$\therefore\ b = \frac{45}{20} = \frac{9}{4}$.

Example. If 6 horses can plough $17\frac{1}{2}$ acres in 4 days, how many acres will 54 horses plough in $2\frac{1}{4}$ days?

Denoting the number of horses by h, of acres by a, and of days by d, we know that $a \propto h$ when d is given, and $a \propto d$ when h is given.

$\therefore\ a \propto dh$;

$\therefore\ a = mdh$.

The first statement gives us $\frac{3}{2}\frac{5}{}=m\times 4\times 6$;

$$\therefore\ m=\frac{3}{4}\frac{5}{8};$$

$$\therefore\ a=\frac{3}{4}\frac{5}{8}dh;$$

$$\therefore \text{ when } d=\frac{9}{4} \text{ and } h=54,$$

$$\text{the number of acres}=\frac{3}{4}\frac{5}{8}\times\frac{9}{4}\times 54=88\frac{1}{3}\frac{9}{2}.$$

Examples. XXXVIII. e.

1. If $y \propto x$, and $y=5$ when $x=6$, find the equation between x and y, and draw its graph.

Find y when $x=9$, and find x when $y=3{\cdot}5$.

Obtain the results graphically and algebraically.

2. If $y \propto \frac{1}{x}$, and $y=3$ when $x=2$, find x when $y=21$.

3. If $y \propto \frac{1}{x^2}$, and $y=5$ when $x=9$, find y when $x=2$.

4. If $a \propto b$ directly and c inversely, and $a=8$ when $b=10$ and $c=15$, find c when $a=1$ and $b=2$.

5. y consists of a constant term and a term which varies as x. $y=3$ when $x=0$, and $y=7$ when $x=1$. Find the equation between x and y, and determine y graphically when $x=3{\cdot}5$.

6. $y \propto x$, and $x \propto \frac{1}{z^2}$; prove $y^{\frac{1}{2}}z=$ a constant.

7. $a^2+b^2 \propto a^2-b^2$; prove that $a+b \propto a-b$.

8. $a+b \propto a-b$; prove that $a^2+b^2 \propto ab$.

9. The attraction of the earth on an external object varies inversely as the square of the distance of the object from the earth's centre. Find the apparent weight of a body at a distance of 2000 miles from the earth's surface, supposing it to weigh 1 lb. on the surface of the earth whose radius may be taken to be 4000 miles.

10. If $xy \propto x^2+y^2$, and 3, 4 be contemporaneous values of x and y, express xy in terms of x^2+y^2.

11. If $y=$ the sum of a constant term and a term varying as xy, and $y=-2\frac{1}{3}$ when $x=2$, and $x=-2$ when $y=1$, express y in terms of x.

12. $x \propto \frac{z}{y^2}$, and $z^2 \propto \frac{y}{x}$; prove $y \propto z \propto \frac{1}{x}$.

13. If $x \propto \frac{1}{y}+\frac{1}{z}$, and $x=3$ when $y=1$ and $z=2$, then $xyz=2(y+z)$.

14. If x varies inversely as $\frac{yz}{y-z}$, and is equal to 5 when $y=7$ and $z=2$, then $xyz=14(y-z)$.

15. The wages of 100 men for 6 months amount to £1080. How many men can be employed for 7 months for £453 12*s.*?

16. With a capital of £450 a man gains £99 in 11 months. What profit does he make in 10 months on a capital of £1000?

17. A garrison of 1500 men has just provisions enough to allow 26 oz. of bread a day to each man for 38 days. The garrison is increased by 400

men. How many ounces of bread must be assigned to each man to prolong the siege for 27 days longer?

18. A sum of money at simple interest amounts to £688 when the rate is $2\frac{1}{2}$ per cent. and the time 3 years. What would be the amount if the rate were $3\frac{1}{4}$ per cent. and the time $2\frac{1}{3}$ years?

19. The pressure of wind on a plane surface $\propto$ the area of the surface and the square of the wind's velocity. The pressure on a sq. foot is 1 lb. when the wind is moving 15 miles an hour. Find the velocity of the wind when the pressure on a sq. yard is 16 lb.

Examples. XXXVIII. f.

Solve algebraically and graphically:

1. The annual expense of a household of 6 persons is £870. Find the expense of 11 persons, supposing £150 of the expense to be constant and the rest to vary as the number of persons.

2. The area of a circle $\propto$ the square of the radius, and the area is 3·14 square metres when the radius is 1 metre. Find the area when the radius is 5 metres. What radius gives an area of 18·84 sq. metres?

3. The distance fallen by a body from rest $\propto$ the square of the time of fall, and a body falls 64 feet in the first 2 secs. How far does it fall in the next 3 secs.?

4. The value of a diamond $\propto$ the square of its weight, and a diamond of 3 carats is worth £8; find the value of one of the same quality weighing 4 carats.

5. y consists of a constant term and a term varying as x. When $x=2$, $y=26$, and when $x=3$, $y=63$. Find y when $x=2{\cdot}5$; and find x when $y=40$.

6. The distance of the horizon at sea $\propto$ the square root of the height of the eye above sea-level. Find the distance when the eye is at a height of 6 feet, given that it is 9 miles when the height is 54 feet. Find the height of the eye when the distance is 4 miles.

Solve algebraically the following:

7. The time of vibration of a pendulum $\propto$ the sq. rt. of its length. The length of one which beats seconds is approximately 39 inches. If it is lengthened by 6 inches, find the time of 1 beat.

8. Weight above the earth's surface varies inversely as the square of distance from the centre, below the surface it varies as the distance from the centre. The earth's radius being reckoned 4000 miles, at what distance below the surface is the weight the same as at 100 miles above it?

9. If a mixture of gold and silver, in which $\frac{3}{4}$ is gold, be worth £49, what will be the value of a mixture of equal weight in which $\frac{1}{2}$ is gold, the value of gold being 16 times that of silver?

10. If the carriages in a railway train be all of the same class and always just full; and if the expense of running a train be proportional to the square of the number of carriages; and if a train of 36 carriages just pay the expense of working it; prove that it will be just as profitable to the railway company to run trains of 16 carriages as trains of 20 carriages.

11. The expenses of a household are partly constant and partly vary as the number of inmates. For 6, 8 and 14 persons the expenses are £16 10s., £18, £22 10s. Draw the graph, find the constant term, and the formula for the expense.

CHAPTER XXXIX.

LOGARITHMS.

238. DEF. **If one number be chosen as base, the logarithm of any number n to this base is the index of the power to which the base must be raised to be equal to n.**

The logarithm of n to the base a is written $\log_a n$.

Thus if $a^p = n$, $p = \log_a n$.

Logarithms calculated to the base 10 are called *common* logarithms.

$10^1 = 10$ $\therefore$ by definition $\log_{10} 10 = 1$,
$10^2 = 100$ $\therefore$ „ $\log_{10} 100 = 2$,
$10^3 = 1000$ $\therefore$ „ $\log_{10} 1000 = 3$.

239. $a^0 = 1$; $\therefore$ $\log_a 1 = 0$, *i.e.*, whatever the base, the logarithm of 1 is zero.

240. *To prove* $\log_a mn = \log_a m + \log_a n$.

Let $\log_a m = x$, and $\log_a n = y$.

Then $m = a^x$, $n = a^y$;

$\therefore$ $mn = a^x . a^y = a^{x+y}$;

$\therefore$ $\log_a mn = x + y = \log_a m + \log_a n$,

i.e. **the logarithm of a product = the sum of the logarithms of its factors.**

Thus $\log 20 = \log 10 + \log 2 = 1 + \log 2$;
$\log 500 = \log 100 + \log 5 = 2 + \log 5$.

241. *To prove* $\log_a \frac{m}{n} = \log_a m - \log_a n$.

Let $\log_a m = x$, $\log_a n = y$.

Then $m = a^x$, $n = a^y$;

$$\therefore \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y};$$

$$\therefore \log_a \frac{m}{n} = x - y = \log_a m - \log_a n,$$

i.e. **the logarithm of a fraction = log numerator − log denominator,** or **the logarithm of a quotient = log dividend − log divisor.**

Thus $\log \cdot 02 = \log \frac{2}{100} = \log 2 - \log 100 = \log 2 - 2$.

242. *To prove* $\log_a n^r = r \log_a n$.

Let $\log_a n = x$, then $n = a^x$;

$$\therefore \quad n^r = (a^x)^r = a^{rx};$$

$$\therefore \quad \log_a n^r = rx = r \log_a n,$$

i.e. **the logarithm of any power of a number is the product of the logarithm of the number and the index of the power.**

Thus $\log 10000 = 4 \log 10 = 4$; $\log 16 = \log 2^4 = 4 \log 2$.

Thus multiplication of numbers is performed by addition of their logarithms, division by subtraction of logarithms, and raising a number to a given power by multiplying the logarithm by the index.

Thus square root is obtained by halving the logarithm and finding the number of which the result is the logarithm.

DEF. If x is the logarithm of n, then n is called the **anti logarithm** of x.

Hence the rule given above takes a simpler form, viz., the square root of a number is obtained by halving its logarithm and finding the antilogarithm of the result.

243. *To prove* $\log_b n = \log_a n \,.\, \log_b a$.

Let $\log_a n = x$; then $n = a^x$.

Take logs of both sides to the base b.

$$\therefore \quad \log_b n = x \log_b a$$

$$= \log_a n \,.\, \log_b a.$$

A particular case of this is

$$\log_a b \,.\, \log_b a = \log_b b = 1.$$

The formula $\log_b n = \log_a n \,.\, \log_b a$ is of great importance; for it enables us to convert logarithms which are calculated to one base into logarithms to any other base.

Example. Find $\log_{10} 2$, given $\log_e 2 = \cdot 693147$, and $\log_{10} e = \cdot 43429$.

$$\log_{10} 2 = \cdot 693147 \times \cdot 43429 = \cdot 3010 \text{ to 4 places.}$$

244. *The integral part of a logarithm is called its* **characteristic** *or* **index,** *the decimal part its* **mantissa.**

$$10^4 = 10000, \; 10^5 = 100000;$$

$\therefore$ the common logarithm of a number between 10,000 and 100,000 is between 4 and 5, and is therefore 4 + a decimal.

$\therefore$ the characteristic of the logarithm of a number of 5 integral figures is 4.

Similarly in other cases.

Thus the logarithm of a number n has for characteristic a number less by 1 *than the number of integral digits in n.*

245. $a^0 = 1$; $\therefore$ $\log_a 1 = 0$.

Thus $\log_{10} 1 = 0$; $\therefore$ the logarithm of a number less than 1 is less than 0, *i.e.* it is negative.

If a logarithm is negative, its mantissa is usually made positive by changing the characteristic.

$$\log_{10} \cdot 3 = \log 3 - \log 10 = \cdot 4771 - 1.$$

This might be written $-\cdot 5339$; but here the mantissa would be negative. The proper way of writing it is $1\cdot 4771$, where the bar over the 1 indicates that the characteristic is negative and the rest positive.

$\log \cdot 0001 = \log \frac{1}{10^4} = \log 10^{-4} = -4$. Similarly $\log \cdot 091 = -$

$\therefore$ the logarithm of any number beginning with a point followed by 3 zeros is between -4 and -3. It must therefore be -4 + a decimal. Thus the characteristic is -4.

A similar argument proves that *when we take the logarithm of any decimal less than* 1, *the characteristic is negative and is numerically one more than the number of zeros before the first significant figure.*

[The significant figures of a number are those which remain when all zeros at the beginning and end have been removed; *e.g.* the significant figures of 0032016 and of 32016000 are 32016.]

It should be noticed that

$$\log \cdot 0032016 = \log \frac{32016}{10^7} = \log 32016 - 7 ;$$

and $\log 32016000 = \log 32016 + \log 1000 = \log 32016 + 3$.

So both these logarithms have the same mantissa as $\log 32016$.

The mantissae of the logarithms of numbers are the same, provided the numbers have the same significant digits.

$$\log \cdot 0003 = \bar{4}\cdot 4771.$$

Suppose it is required to multiply this by a number, say 5.

$$\bar{4}\cdot4771 \times 5 = (-4 + \cdot4771) \times 5 = -20 + 2\cdot3855 = \bar{18}\cdot3855.$$

$$\frac{\bar{4}\cdot4771}{5} = \frac{-5 + 1\cdot4771}{5} = -1 + \cdot2954 = \bar{1}\cdot2954.$$

Examples. XXXIX. a.

(*Except in question* 2 *the logarithms are to the base* 10.)

1. Find $\log 1000$, $\log 100{,}000$, $\log 1$, $\log \cdot01$ and $\log \cdot0001$ to the base 10.

2. Find $\log 64$, $\log 256$, $\log 1$, $\log \frac{1}{2}$, and $\log \frac{1}{16}$ to the base 2.

3. Find the value of $2\log 3 + 2\log 9 + \frac{2}{3}\log 27 - 2\log 81$.

4. Given $\log 2 = \cdot3010$, find $\log 64$, $\log 3\cdot2$, $\log 1\cdot28$, and $\log \frac{1}{2}$ to 4 places of decimals.

5. Given $\log 3 = \cdot4771$, find $\log 243$, $\log 24\cdot3$, $\log 810$ and $\log \cdot081$ to 4 places.

6. Given $\log 2$, find $\log 5$, $\log 500$, $\log \cdot4$, and $\log 125$ to 4 places.

7. Given $\log 2$ and $\log 3$, find $\log 576$, $\log 1728$ and $\log \cdot864$ to 4 places.

8. Find the characteristic of $\log 317$, of $\log 1234$, of $\log 12\cdot3$, of $\log \cdot623$ and of $\log \cdot048$.

9. Find to 3 places by logarithms $\sqrt[4]{3}$, and check the result by finding the square root of the square root of 3.

10. Given that $a = 10^5 \cdot b$, find the difference between $\log a$ and $\log b$.

11. Find $10^{\cdot301} \div 10^3$, given that $\log 2 = \cdot3010$.

Examples. XXXIX. b. (*Chiefly Oral.*)

Find

1. $\log_9 81$, $\log_4 16$, $\log_3 27$, $\log_5 125$, $\log_4 64$.

2. $\log_7 343$, $\log_2 32$, $\log_2 8$, $\log_{10} 1000$, $\log_3 729$, $\log_3 81$.

3. $\log_9 3$, $\log_{27} 3$, $\log_3 243$, $\log_5 3125$, $\log_{25} 5$, $\log_{125} 5$.

4. $\log_8 16$, $\log_4 32$, $\log_{16} 8$, $\log_8 2$, $\log_{\frac{1}{2}} 8$, $\log_{\cdot01} 10$.

5. Give other forms of the logarithms of abc, $a^2b^3c^5$, x^5yz^2, $\frac{ab^2}{cd^3}$.

6. Find $\log 4$, $\log 6$, $\log 8$, $\log 9$, $\log 15$, $\log 18$ in terms of $\log 2$ and $\log 3$ to the base 10.

7. Simplify $\log 2 + \log 5$, $\log 2 + \log 3$, $\log 12 - \log 3$, $\log 12 - 2\log 2$, $\log 54 - 2\log 3$, $\log 3000$.

Solve the equations

8. $2^x = 256$. **9.** $2^x \times 3^x = 216$. **10.** $5^x = 3125$.

11. $2^x = \cdot5$. **12.** $2^x = \cdot125$.

13. In the logarithms of 6043, 32, 801, 89567, ·0017, ·32, 3·241, ·0000386, 56·24 to the base 10, give the characteristic.

The student is supposed to have a table of logarithms. The following are quoted because they are of very frequent occurrence :

$$\log 2 = \cdot3010, \quad \log 3 = \cdot4771, \quad \log 7 = \cdot8451.$$

Examples. XXXIX. c.

1. Explain why $\log 3251$ and $\log 9999$ have the same characteristic
2. Prove that $\log 723$ and $\log 7{\cdot}23$ have the same mantissa.
3. In what other form can $\log(a^2-b^2)$ be put?
4. Find $\log 9^2 - \log 3^2$ and $\log(9^2-3^2)$.
5. Add together $1{\cdot}2864$, $2{\cdot}1572$, $\bar{3}{\cdot}7632$.
6. Add together $\bar{5}{\cdot}6391$, $1{\cdot}5534$, $\cdot 7431$.
7. Subtract $1{\cdot}2345$ from $\bar{2}{\cdot}6387$.
8. Subtract $\bar{2}{\cdot}5461$ from $1{\cdot}0386$.
9. Multiply $\bar{2}{\cdot}4771$ by 5.
10. Multiply $\bar{3}{\cdot}6990$ by 2.
11. Divide $\bar{1}{\cdot}5423$ by 3.
12. Divide $\bar{2}{\cdot}3184$ by 4.
13. Divide $\bar{3}{\cdot}3075$ by 5.
14. How many zeros follow the decimal point in $\cdot 2^{20}$?
15. Find $\log 2592$, given $\log 2$ and $\log 3$. Hence find $\log 2{\cdot}592$, $\log 259200$, $\log \cdot 002592$.
16. How many digits are there in 2^{20}?
17. ..... 2^{25}?
18. What are the significant figures in 203500, $20{\cdot}35$, $\cdot 00176$, 1760, $\cdot 01308$?
19. Give the characteristic of $\log 1760$, of $\log 1{\cdot}76$ and of $\log \cdot 00176$.
20. Calculate these logarithms from $\log 2$ and $\log 11$.
21. Find $\log 5^3 . 3^4 . 7^2$, given $\log 2$, $\log 3$, $\log 7$.
22. Find the square roots of 3, 7, 2·5, 11 graphically by means of the curve $y=x^2$.
23. Find them also by logarithms.
24. Simplify $\frac{1}{2}\log 196 + \log 25 - 2\log 2 + \log \frac{3}{7}$.
25. Find $\log 3528$ from $\log 2$, $\log 3$ and $\log 7$.
26. Hence find $\log 35{\cdot}28$, $\log 352700$ and $\log \cdot 03528$.
27. How many digits are there in 7^{30}?
28. Express 3, 4, 5 and 105 as powers of 10.

Napierian Logarithms.

246. Logarithms were originally calculated by their inventor, Napier, to the base e, where e is equal to the series

$$1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\ldots \text{ to infinity,}$$

and to that base they are used in theoretical investigations. Such logarithms are converted into those to the base 10 by multiplying by $\frac{1}{\log_e 10}$, *i.e.* by $\cdot 43429\ldots$.

Common Logarithms.

247. In arithmetical operations, including the numerical results of solution of triangles in Trigonometry, *common logarithms*, as those to the base 10 are called, are employed.

It is natural that a beginner in the use of logarithms, who is constantly employing the information that $\log_{10}2 = \cdot3010$, $\log 3 = \cdot4771$, $\log 7 = \cdot8451$, etc., should feel anxious to obtain a verification of these statements for himself before he can calculate them by the series which involve the quantity e. By the following simple methods he may satisfy himself that they are at all events approximately true.

By taking the square root of 2, and the square root of the result, and so on, we can obtain the results given below:

$$2^{\frac{1}{2}} = 1\cdot4142135, \quad 2^{\frac{1}{4}} = 1\cdot189207, \quad 2^{\frac{1}{8}} = 1\cdot0905,$$

$$2^{\frac{1}{16}} = 1\cdot044, \quad 2^{\frac{1}{32}} = 1\cdot022, \quad 2^{\frac{1}{64}} = 1\cdot011,$$

$$2^{\frac{1}{128}} = 1\cdot0055, \quad 2^{\frac{1}{256}} = 1\cdot0028,$$

$$2^{\frac{9}{256}} = 2^{\frac{1}{256}} \times 2^{\frac{1}{32}} = 1\cdot024 \text{ nearly} = 2^{10} \div 10^3 \text{ nearly};$$

$$\therefore \tfrac{9}{256}\log 2 = 10\log 2 - 3; \quad \therefore \log 2 = \tfrac{96}{319} = \cdot301.$$

More general results may be obtained by calculating $10^{\frac{1}{2}}$, $10^{\frac{1}{4}}$, $10^{\frac{1}{8}}$, etc., in the same way.

Thus we get

$$10^{\frac{1}{2}} = 3\cdot1622776,$$

$$10^{\frac{1}{4}} = 1\cdot77828,$$

$$10^{\frac{1}{8}} = 1\cdot3335,$$

$$10^{\frac{1}{16}} = 1\cdot155,$$

$$10^{\frac{1}{32}} = 1\cdot074.$$

By Multiplication we get $10^{\frac{3}{32}}$, $10^{\frac{5}{32}}$, $10^{\frac{7}{32}}$, and thus we have $10^{\frac{1}{32}}$, $10^{\frac{2}{32}}$, $10^{\frac{3}{32}}$, ... $10^{\frac{31}{32}}$.

Taking logarithms of both sides of each statement we obtain

$$\log 1\cdot074 = \tfrac{1}{32} = \cdot03125,$$

or

$$\log 1074 = \log(1\cdot074 \times 10^3) = 3 + \log 1\cdot074 = 3\cdot03125,$$

$$\log 1155 = 3 + \log 10^{\frac{1}{16}} = 3\cdot0625,$$

$$\log 1239 = 3 + \log 10^{\frac{3}{32}} = 3\cdot0938,$$

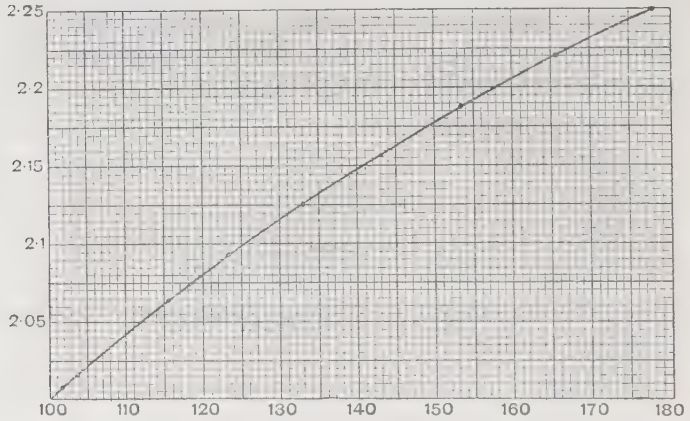
$$\log 1333 = 3 + \log 10^{\frac{1}{8}} = 3\cdot1250.$$

These logarithms will be found, on comparison with tables of 4-figure logarithms, to be generally correct to 3 places.

More exact results may be obtained by calculating $10^{\frac{1}{64}}$, $10^{\frac{2}{64}}$, etc.

Graphic Representation.

248. Take a piece of squared paper, ruled in inches and tenths of an inch. Let the abscissae represent the numbers, $\frac{1}{10}$ of an inch representing unity; and let the ordinates represent the logarithms, 1 inch representing ·1.



Take an intersection of lines near the bottom left hand corner to represent (100, 2), and mark the successive inches horizontally as 110, 120, 130, etc. Mark the successive inches vertically as 2·05, 2·1, 2·15, etc.

$\log 1{\cdot}018 = \log 10^{\frac{1}{128}} = \frac{1}{128} = {\cdot}008$; $\therefore \log 101{\cdot}8 = 2{\cdot}008$;

$\log 1{\cdot}036 = \log 10^{\frac{1}{64}} = \frac{1}{64} = {\cdot}016$; $\therefore \log 103{\cdot}6 = 2{\cdot}016$;

$\log 107{\cdot}4 = 2{\cdot}031$;

$\log 115{\cdot}5 = 2{\cdot}063$;

$\log 123{\cdot}9 = 2{\cdot}094$;

$\log 133{\cdot}4 = 2{\cdot}125$;

$\log 143{\cdot}2 = 2{\cdot}156$, since $10^{\frac{5}{32}} = 1{\cdot}432$;

$\log 153{\cdot}8 = 2{\cdot}188$, since $10^{\frac{6}{32}} = 1{\cdot}538$;

$\log 165{\cdot}2 = 2{\cdot}219$, since $10^{\frac{7}{32}} = 1{\cdot}652$;
$\log 177{\cdot}8 = 2{\cdot}250$.

Mark the points whose abscissae are the numbers and whose ordinates are the corresponding logarithms, and connect them by a curve.

By means of this curve we can write down approximately the logarithms of any intermediate numbers.

From the approximate straightness of the curve, where small lengths of it are considered, we conclude that, for numbers differing by small quantities, *i.e.* by small fractions of themselves, the differences of the logarithms are nearly proportional to the differences of the numbers.

249. This principle, **the Principle of Proportional Parts** as it is called, is most important in the construction of tables.

As we know that $\log 133{\cdot}3 = 2{\cdot}125$
and $\log 143{\cdot}2 = 2{\cdot}156$,
we may roughly approximate to $\log 135$.

Here the numbers differ by 9·9, and their logarithms differ by ·031.

If the numbers differed by $\frac{1}{3}$ of 9·9, their logarithms would differ by $\frac{1}{3}$ of ·031; and similarly for other cases.

If we wish to obtain $\log(133{\cdot}3 + x)$ from $\log 133{\cdot}3$, we may call it $2{\cdot}125 + y$, where we recognize that y is the same fraction of ·031 as x is of 9·9.

Thus $\log 135 = \log(133{\cdot}3 + 1{\cdot}7) = 2{\cdot}125 + y$

where $$\frac{y}{{\cdot}031} = \frac{1{\cdot}7}{9{\cdot}9},$$

i.e. $$y = \frac{1{\cdot}7 \times {\cdot}031}{9{\cdot}9} = {\cdot}005;$$

$$\therefore \log 135 = 2{\cdot}125 + {\cdot}005 = 2{\cdot}13.$$

This result is obtained as an illustration of the principle: but it is only a rough approximation, since the difference between the numbers chosen is not small enough to make it accurate. The same result might be obtained by reading it off from the graph described.

250. MATHEMATICAL TABLES.

Logarithms.

	0	1	2	3	4	5	6	7	8	9	1 2 3	4 5 6	7 8 9
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3 6 8	11 14 17	20 22 25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3 5 8	11 13 16	18 21 24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2 5 7	10 12 15	17 20 22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2 5 7	9 12 14	16 19 21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2 4 7	9 11 13	16 18 20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2 4 6	8 11 13	15 17 19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2 4 6	8 10 12	14 16 18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2 4 6	8 10 12	14 15 17

The quotation given above from a table of 4-figure logarithms will show the method of reading the logarithm of any number. For instance suppose that $\log 1756$ is required. Look along the line beginning 17 until you reach the figures below the 5 which occurs in the top line. The figures are 2430. For the final 6 take the figures below 6 in the columns on the right of the page, viz. 15. The total result is 2445. In reality the figures are ·2430 and ·0015, giving a total of ·2445 as the mantissa of the logarithm of a number whose significant figures are 1756.

Add the proper characteristic, and the logarithm of 1756 is 3·2445.

Also $\log 17{\cdot}56 = 1{\cdot}2445$ and $\log {\cdot}01756 = \bar{2}{\cdot}2445$;

for the logarithms of numbers have the same mantissa if the numbers themselves have the same significant digits.

By the principle of proportional parts if we required $\log 1756{\cdot}7$ we should have to add $\frac{1}{10}$ of the difference for 7; *i.e.* $\frac{{\cdot}0017}{10}$, *i.e.* ·0002.

Thus $\log 1756{\cdot}7 = 3{\cdot}2447$.

Antilogarithms.

251. The reverse process, that of finding the antilogarithm of a set of figures (*i.e.* the number whose logarithm is the given set of figures), can be accomplished by searching in the columns for the given set of figures or the set next less than these, and making, in the latter case, the proper allowance for the difference by means of the right-hand columns. Labour is saved, however, by using

tables of **antilogarithms**, which are read in a similar manner to the tables of logarithms.

It must be remembered that the mantissa only is given in the table and only the **significant digits** of the antilogarithm. The position of the decimal point in the antilogarithm must be determined by the given characteristic.

Antilogarithms.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4

Find the antilogarithm of ·2445 and of 3·2445.

The significant digits are found by adding to 1754 the figure under 5 in the right-hand columns.

Thus the antilogarithm of ·2445 is 1·756, since the significant digits are 1756 and the characteristic 0 shows that there is *one* integral figure.

So also 3·2445 is the logarithm of 1756.

252. The tables of logarithms may be used in such a manner as to obtain the proper characteristic without direct reference to the rules given for writing down the characteristic. The method itself explains how the characteristic occurs. Any logarithm as it appears in the tables, viz. with 0 for characteristic, is the logarithm of a number containing one integral digit. This may be regarded as the **standard form**; and all numbers, whose logarithms are required, may be expressed in terms of this standard form by multiplying or dividing by some power of 10.

For instance $\log 7{\cdot}253 = 0{\cdot}8605$.

$$\log 7253 = \log (7{\cdot}253 \times 10^3) = {\cdot}8605 + 3 = 3{\cdot}8605.$$

$$\log {\cdot}0007253 = \log (7{\cdot}253 \times 10^{-4}) = {\cdot}8605 - 4 = 4{\cdot}8605.$$

Examples. XXXIX. d.

Reduce each of the following (in questions 1 to 5) to *standard form* and find their logarithms. Check the results by the table of antilogarithms.

1. 36840 and 368·4.
2. 1567 and ·01567.
3. 428·6 and 4286000.
4. 2113·5 and ·0021135.
5. 386·5, ·05713, 7641000, ·76483.
6. Find approximately by logarithms the product of 1·414 and 1·732.
7. ..... $\sqrt{7}$.
8. ..... $\sqrt[3]{7}$.
9. Solve the equation $2^x = 3$.
10. Solve the equation $3^{x+4} = 405$.
11. Solve $3^x \cdot 2^y = 100$, $2^x \cdot 3^y = 50$.
12. Find $\log_2 32$, $\log_5 125\sqrt{5}$, $\log_{25} 5\sqrt{5}$. Find $\log 1728$ to the base $2\sqrt{3}$.
13. Find the characteristics of $\log_3 85$, $\log_4 71$, $\log_2 1000$.
14. Plot the curve $y = 2^x$ from $x = 0$ to $x = 6$, using 1 inch for the unit of abscissae, one-tenth for unit of ordinates. Hence find $2^{1\cdot5}$, $2^{3\cdot4}$, $2^{4\cdot5}$.

Find also $\log_2 14$ and $\log_2 23$.

15. From $\log 2$, $\log 3$ and $\log 7$ find $\log_{10} 252$: and hence $\log 25{\cdot}2$, $\log 25200$, $\log {\cdot}252$ and $\log {\cdot}00252$.
16. Find the common logarithms of 3456, ·0625 and 245 by means of the factors.
17. Simplify $\log 98^{\frac{1}{2}} + \log 245 - \log 17{\cdot}15$.
18. Simplify $\log 15 + \log 105 - \log 225$.
19. Find $\log 81$ to the base 243 and $\log 125$ to the base 2·5.
20. Prove $\log 20 + 7 \log \frac{1}{16} + 5 \log \frac{24}{25} + 3 \log \frac{80}{81} = 1$.

Find approximately the results of the following by four-figure logarithms:

21. $2{\cdot}3 \times 1270$.
22. $\cdot2413 \times 6{\cdot}052$.
23. $4{\cdot}951 \times 2{\cdot}836$.
24. $\cdot3463 \times \cdot3973$.
25. $\cdot746 \times \cdot6235$.
26. $3{\cdot}72^4$.
27. $\cdot407 \times 40{\cdot}3 \times \cdot006$.
28. $\cdot0438 \times 937$.
29. $48{\cdot}25 \div 634{\cdot}9$.
30. $\cdot07644 \div 147$.
31. $\cdot86751 \div 24{\cdot}3$.
32. $8{\cdot}30676 \div 3596$.
33. $\sqrt{184{\cdot}12}$.
34. $\left(\cdot9375 \times \cdot018^{\frac{1}{7}}\right)^{\frac{1}{11}}$.
35. $\sqrt{\dfrac{15{\cdot}92^{\frac{1}{7}} \times \cdot0182^{\frac{1}{3}}}{\cdot00526 \times \overline{196}^5}}$.
36. $\sqrt{10}$ correct to 3 places.
37. $3{\cdot}8^3$.
38. $\sqrt[3]{68921}$.
39. $\dfrac{1}{\sqrt{\cdot0714}}$.
40. $\dfrac{1}{\sqrt{\cdot0196}}$.
41. Solve $4 \times 2^{2-x} = 5^{6x-4}$.

42. By successively taking square roots it may be found that $10^{\frac{1}{2}} = 3{\cdot}1623$, $10^{\frac{1}{4}} = 1{\cdot}778$, $10^{\frac{1}{8}} = 1{\cdot}334$, $10^{\frac{1}{16}} = 1{\cdot}155$, $10^{\frac{1}{32}} = 1{\cdot}074$.

By multiplication obtain $10^{\frac{3}{32}}$, $10^{\frac{5}{32}}$.

Plot the curve $y = 10^x$, using 16 inches for the unit of x and 2 inches for that of y. From this read off $\log_{10} 1{\cdot}5$ and $\log_{10} 2$.

43. Find by logarithms $\frac{6}{7}\frac{6}{8}\frac{9 \times 3}{2}\frac{7}{}$ correct to 2 decimal places.
44. Prove that $\log_y \left(\dfrac{x}{y^z}\right) = \log_y x - z$.

45. If $x = \log_b a$, what is the log of a^n to the base b^n?

46. By the aid of tables prove that approximately

$$200^{\frac{1}{10}} + \log_{10} \cdot 2003 = 1.$$

47. Compute by tables $\frac{2}{2}\frac{5}{7} \sqrt[3]{34 \cdot 3} \div \sqrt{14 \cdot 4}$.

48. If $\log \frac{1}{1}\frac{0}{0}\frac{2}{2}\frac{5}{4} = a$, and $\log 2 = \beta$, show that $\log 4100 = a + 12\beta$.

49. Prove that $\log 1250 + \log \cdot 343 - \log 49 + \log 25 + \log 6 \cdot 4 = 2 + \log 14$.

50. Find x correct to 2 decimal places, if $5^x \cdot 7^{x-1} = 11^{x+2}$.

51. Simplify $\log \cdot 06 + \log (\cdot 6)^2 - \log 4 - \log 54 + 4$.

52. If the logarithm of a given number to base 4 is ·3518, what is its logarithm to the base 8?

53. Given $5^{x+2} = 8^{2x-1}$, find x correct to 3 places.

54. Calculate the following by logarithms and show how you would roughly check your results:

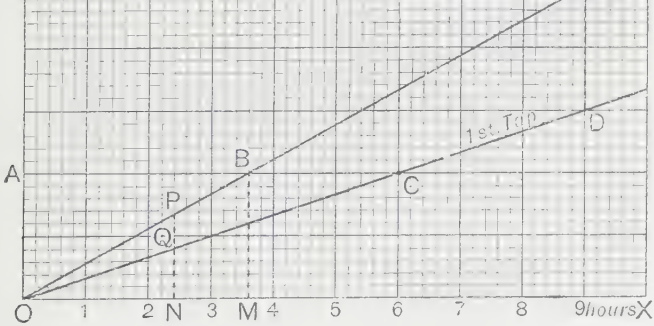
(i) pr^n, where $p = 93 \cdot 75$, $r = 1 \cdot 03$, $n = 4$;

(ii) $\frac{4}{3}\pi r^3$, where $\pi = \frac{3}{1}\frac{5}{1}\frac{5}{3}$, $r = 5 \cdot 875$.

CHAPTER XL.

HARDER GRAPHICAL AND MISCELLANEOUS PROBLEMS.

253. *One tap will fill a cistern in 6 hours; a second will fill it in 9 hours: how long will they take to fill the cistern, running together?*



Measure time horizontally along OX as shown in the diagram. [One-half in. (reduced) = 1 hour.]

Let OA (1 in.) measured vertically along OY denote the capacity of the cistern. OC is the graph of the first tap.

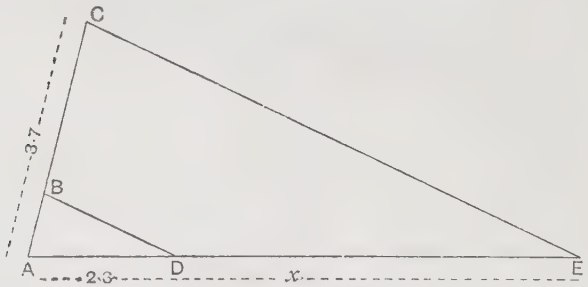
From D, the pt. where this meets the vertical 9 hour line, take DE = 1 in. upwards.

OE is the graph of the work done by the two taps.

Thus PN is the portion of the cistern filled by the two taps, running together, in time ON.

BM = OA. ∴ OM is the reqd. time = 3·6 hours.

254. *Multiply* 3·7 *by* 2·3 *by a graphical method.*



Draw two str. lines AC, AE. With a convenient unit, make AB = 1, AC = 3·7, AD = 2·3.

Join BD and draw CE ‖ to BD.

Then $\frac{AE}{AD} = \frac{AC}{AB}$, *i.e.* AE × unity = AD × AC

= 2·3 × 3·7.

∴ the number of units of length in AE represents the product 2·3 × 3·7.

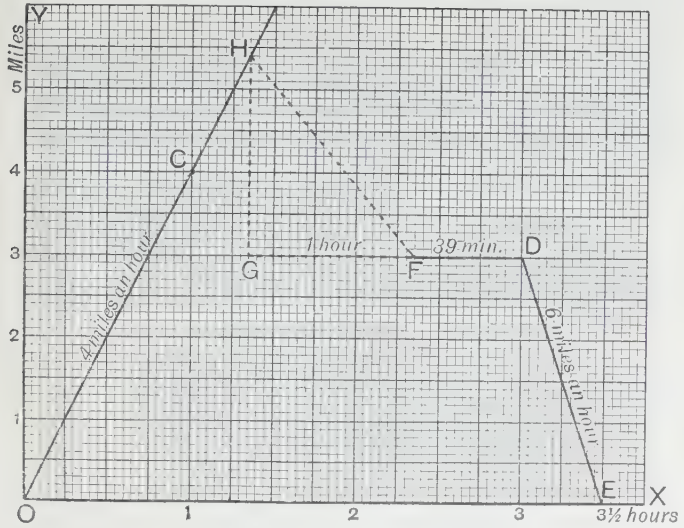
By measurement, AE = 8·51, the reqd. product.

Squared paper might be used for the above, BAD being made a rt. angle.

Division may be performed in a similar manner.

255. *A man walked from* A *to* B *at the rate of* 4 *miles an hour, without delay ran back at the rate of* 6 *miles an hour for half an hour, then waited for* 39 *minutes, and then completed the journey home, by*

walking the rest of the way, in an hour. How far is it from A *to* B, *if the whole journey occupied* $3\frac{1}{2}$ *hours, and what distance did he walk in the last part of the journey?*



[In the printed diagram, which is reduced, the side of each small square denotes one-tenth of an inch.]

Measure time horizontally along OX, taking 2 in. to denote an hour, and distance vertically along OY, taking 1 in. to represent a mile.

OCH is the graph of 4 m. an hour, DE is the graph of the distance he runs at 6 m. an hour, DF = 39 minutes, the time he waits.

FG = 1 hour. Let the vertical line through G meet OC at H.

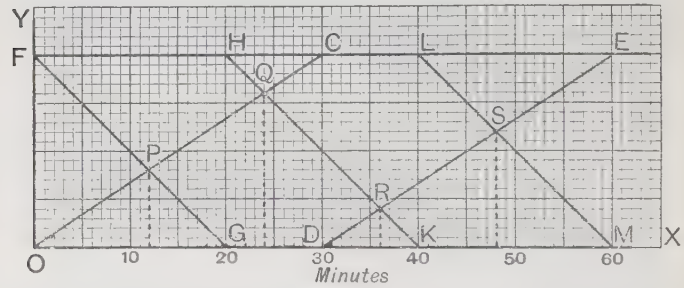
Then HF is the graph of his walk at the end of the journey.

Thus we see that the vertical distance of H from OX = the distance of A from B. ∴ from the diagram, the dist. reqd. = 5·4 miles.

Also GH = 2·4 miles = the dist. he walked at the end of the journey.

N.B.—The order in which the different parts of the journey and the waiting are taken is immaterial.

256. A *walks at the rate of* 2 *miles an hour,* B *at the rate of* 3 *miles an hour, round a circular track* 1 *mile long, starting at the same point, and at the same time in opposite directions. Find the times of their first four meetings.*



The point to observe is, that between each two meetings the men together walk one mile.

Measure times horizontally along OX as shown in the diagram

Take OF vertically along OY to denote one mile.

OC is A's graph for the first 30 minutes (1 mile).

A is then at the starting point. ∴ we may take DE for his graph in the next 30 min.

Taking F as B's starting point, FG is his graph for the first 20 min., and as with A, HK his graph for the second 20 min. and LM his graph for the third 20 min.

The points P, Q, R, S, where these meet, give us the times reqd., which are 12, 24, 36, 48 minutes from the start.

257. *A man receives* 4 *shillings for every day that he works, but is fined* 1*s.* 6*d. for every day he is absent. After* 25 *days he receives* £2. 16*s. in wages. How many days was he absent?*

Take OA to represent 25 days, OB to represent 100 shillings, so that OC is the graph of the money the man earns.

Draw AD, the graph of his fines, from the point A, instead of from O.

Examining any ordinate PQN, we see that

PN represents the money he earns in ON days,

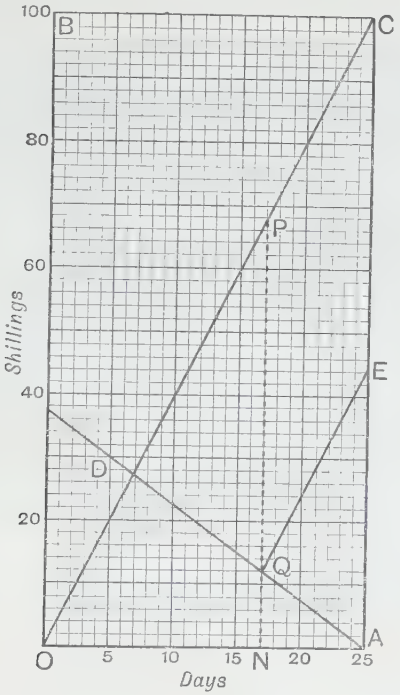
and QN ..... amount of his fines in AN days;

$\therefore$ PQ represents the money he actually receives.

To solve this problem we have to draw PQ equal to 56 shillings.

Take CE equal to 56 shillings, and draw EQ parallel to OC to meet AD at Q. Draw the ordinate PQN.

$$PQ = EC = 56;$$



$\therefore$ ON represents the number of days he is at work,
and AN the number of days he is absent.

From the diagram we see that he is absent from work for 8 days.

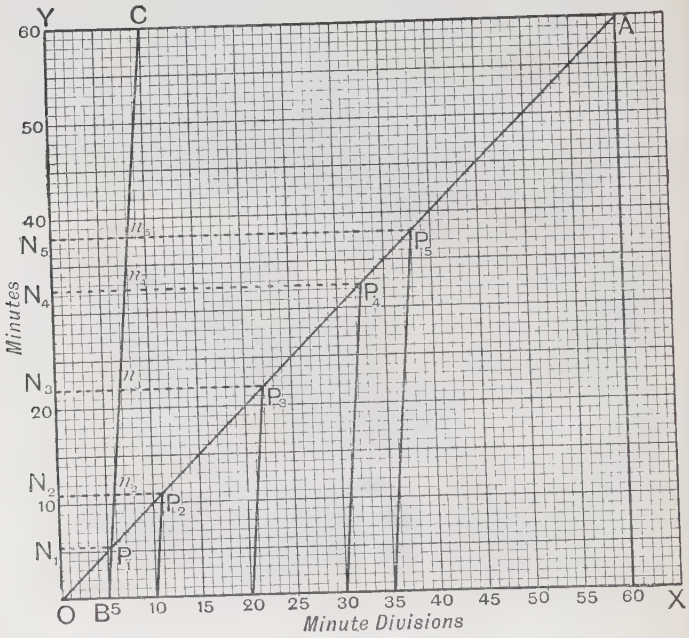
258. *Find graphically (to the nearest minute) the times after* 1 *o'clock when the hands of a clock are* (1) 5 *minute divisions apart,* (2) *first at right angles,* (3) 25 *minute divisions apart,* (4) *pointing in opposite directions.*

Measuring minute divisions horizontally and minutes of time

vertically, as shown in the diagram, OA is the graph of the minute hand.

Remembering that the hour hand starts at 1 o'clock, and turns through 5 minute divisions in an hour, BC is its graph.

Drawing through the 10, 20, 30, and 35 minute division points straight lines parallel to BC, we obtain the points P_2, P_3, P_4, P_5.



Through these points draw parallels to OX.

$P_2m_2 = 5$ minute divisions. $\therefore$ ON_2 gives the time when the hands are 5 minute divisions apart.

Similarly

ON_3 gives the time when the hands are first at rt. angles,

ON_4 ..... 25 min. divisions apart,

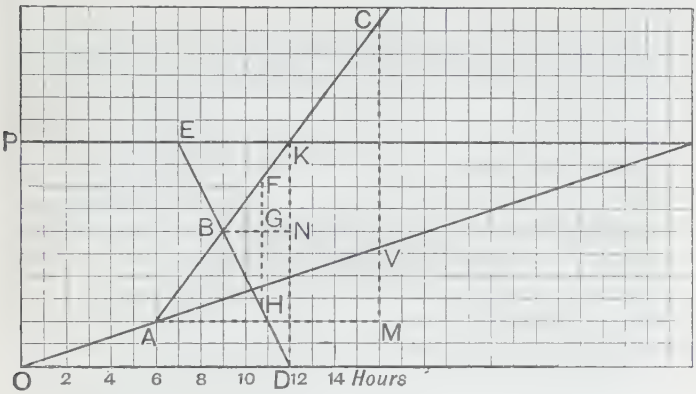
ON_5 ..... pointing in opp. directions.

The times, to the nearest minute, are

(1) 11 min. past one. (2) 22 min. past one.

(3) 33 ..... (4) 38 .....

259. *A tap, which would fill a cistern in* 30 *hours, is set running; after* 6 *hours, an additional tap, which running alone would fill the cistern in* 10 *hours, is opened. After another* 3 *hours, a valve is opened and with the two taps still open, the cistern is emptied in* 3 *hours more. How long would the valve take to empty the full cistern if the taps were shut?*



Measure the hours horizontally as shown in the diagram, and let OP, drawn vertically, represent the capacity of the cistern.

OA is the graph of the first tap.

From A, where it cuts the 6 hour line, measure AM equal to 10 hours, and draw the ordinate MVC meeting OA at V.

Take VC equal to OP, the capacity of the cistern, and join AC. AC is the graph of the work done by the two taps running together. Take the point B on this graph, 3 hours from A.

BD is the graph of the work done by the two taps and the valve.

Examining any ordinate FGH between BC and BD, we see that GF = the part of the cistern filled by the taps in time BG, and therefore FH = the part of the cistern emptied by the valve in the same time.

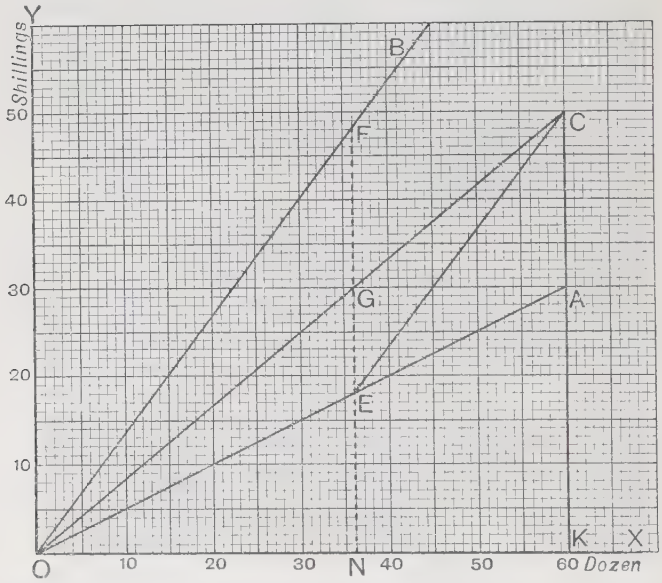
Hence to find the time taken by the valve alone to empty the full cistern, we have to find the points where the vertical distance between BC and BD is equal to OP the capacity of the cistern.

KD is this line. Therefore BN denotes the time taken by the valve alone to empty the full cistern.

From the figure BN = 3 hours.

This should be verified by the algebraic method.

260. *A man mixes wine at* 30*s. a dozen with wine at* 80*s. a dozen. How many dozen of each kind must he take in order that a mixture of* 60 *dozen may be worth* 50*s. a dozen?*



In the diagram, OA is the graph of the 30*s.* wine, OB of the 80*s.* wine, OC of the mixture.

From C, the pt. (60, 50), draw CE parallel to OB to meet OA at E.

Draw the ordinate FGEN as shown.

ON (= 36) gives the no. of dozen of the cheaper kind.

NK (= 24) ..... dearer kind.

Proof. The gain on ON dozen of the cheaper is CA × ON, *i.e.* GE × 60.

∴ if he sold ON dozen of the cheaper wine at 50*s.* GE × 60 would represent his gain.

If he sold ON dozen of the dearer wine at 50*s.* FG × 60 would represent his loss.

Now $$\frac{ON}{NK} = \frac{OG}{CG} \text{ (similar } \triangle\text{s ONG, OKC)}$$

$$= \frac{FG}{GE} \text{ (..... OFG, CGE);}$$

$$\therefore\ GE = FG \times \frac{NK}{ON};$$

i.e. his gain on ON dozen of the cheaper = his loss on NK dozen of the dearer wine, when he sells the mixture at 50*s.*

∴ he sells 36 dozen of the cheaper with 24 dozen of the dearer.

Examples. XL. a.

The following problems should be solved by graphical methods. They may be verified algebraically.

1. One tap will fill a cistern in 6 hours, and a second will fill it in 4 hours. How long will they take to fill the cistern running together?

2. A tap would fill a cistern in 10 hours. When the cistern is half full a second tap is turned on and the cistern is full in one hour more. How long would the second tap, running alone, take to fill the cistern?

3. A man can dig over a garden in 5 days, and a boy in 10 days. What fraction of the garden do they dig over working together for 2 days?

4. A man starts digging a garden at a rate which would complete the work in 10 days. After working for 4 days, he takes 2 days' rest, and after that, working at a faster rate still does the work in the 10 days. Find approximately how long he would take to do all the garden working at the faster rate.

5. A does in 5 days a piece of work which B does in 7 days. For how many complete days must they be engaged to do the work when they work together?

6. A cistern has 3 pipes A, B, C; A and B can fill it in 4 and 5 hours respectively, and C can empty it in 2 hours. If these pipes are opened in order at noon, 1 o'clock, and 2 o'clock, find when the cistern will be empty.

7. A walks round a circular track one mile long in 20 minutes, and B motors round it in 5 minutes in the opposite direction, but starting from the same point. Draw graphs to shew when and where they meet, distances to be measured in A's direction.

8. A travels at the rate of 7 miles an hour, and B at 2 miles an hour round a circular track one mile long, starting at the same time from the same point in the same direction. Find the first three times when A passes B. Also write down the distances from the start, measured in the direction of the travellers, at which the passing points lie.

9. 1000 German marks are equal to £49. Find graphically the value of 550 marks in pounds, and £65 in marks, to the nearest pound, and 10 marks respectively.

10. Find the square root of $2{\cdot}3 \times 3{\cdot}5$.

11. In a hundred yards race, A beats B by 11 yards, and C by 19 yards. Find to the nearest yard how much B beats C by in 100 yards. (Use millimetre paper.)

12. A walks at the rate of 4 miles an hour, resting for half an hour at the end of each hour. B, starting 3 hours later, and travelling uniformly without resting, catches him up 14 miles from home. Draw their graphs, and determine B's pace.

13. If I buy oranges at the rate of 5 for 3*d.*, how much, to the nearest penny, shall I give for 57?

14. Two trains start at the same time, one from Liverpool to Manchester, and the other from Manchester to Liverpool, and running steadily complete the journey in 42 minutes and 56 minutes respectively. How long is it from the moment of starting before they meet?

15. A and B run over a course, B having 50 yards start. A runs 4 yards while B runs 3 yards, and arrives at the end of the course 150 yards ahead of B. Find the length of the course.

16. Two men run at uniform rates over a course of 4000 yards. One starts 24 seconds after the other and arrives 16 seconds before him. Where does he pass him?

17. Suppose that it takes a train just 6 days to run the whole length of the road, and that one train leaves each end of the road each morning. How many trains will a person meet going the length of the road, not counting the train which arrives just as he starts, nor the train which starts just as he arrives?

18. A man receives 3*s.* 6*d.* for every day that he works, but is fined 1*s.* for every day he is absent. After 20 days he receives the same wages that he would have earned by working steadily for 11 days. How many days was he absent from work?

19. A, walking from P to Q at the rate of 4 miles an hour, starts one hour before a coach travelling at the rate of 12 miles an hour, and is picked up by the coach. On arriving at Q he finds that his coach journey has lasted 2 hours. Find the distance from P to Q.

20. A, B and C travel from the same place at the rates of 4, 5 and 6 miles an hour respectively, and B starts 2 hours after A. How long after B must C start in order that they may overtake A at the same instant?

21. A man walked from A to B at the rate of 3 miles an hour, bicycled back without delay for 2 miles at the rate of 8 miles an hour, and walked the remaining distance home at the rate of 2 miles an hour, taking 4 hours over the journey. How far is it from A to B?

22. A man walks from A to B at the rate of 4 miles an hour, waits half an hour at B, and then returns at the rate of 3 miles an hour. He took $3\frac{1}{2}$ hours over the journey. Find the distance from A to B.

23. A man did a journey of 11 miles in the following manner. He walked part of the way, then bicycled at the rate of 11 miles an hour, and finally completed the journey by walking for 15 minutes. If his walking was at the rate of 3 miles an hour, and he took 3 hours over the journey, find how far he bicycled.

24. A man walked a certain distance at the rate of 4 miles an hour, and then ran part of the way back at the rate of 6 miles an hour, walking the remaining distance home in 15 minutes. The whole journey took him

1 hour and 20 minutes. How far did he run, and what is the distance? If a second man does the reverse double journey uniformly, starting at the same time, and occupying the same time, when and where will they meet?

25. Two towns are 50 miles apart: A is to leave one of these towns at 6 o'clock and to arrive at the other at noon, making four stops of half an hour each at 10, 20, 30 and 40 miles from the starting point. B leaves the other end of the road at 7 o'clock, travels 20 miles an hour for an hour, then turns back and retraces his course for an hour at the rate of 10 miles an hour, then turns round and advances again at such a rate as to meet A as he is starting from his third halt: continuing at the same rate B meets at 10.30 a third man C, who left the first end of the route 2 hours later than A, and has been going at a uniform rate. At what rate has C been travelling, and where did B meet him?

26. A travels 6 miles in the first hour, 5 in the second, 4 in the third, and so on. B starting 14 miles behind him travels at the uniform rate of 7 miles an hour. When and where does he catch B up?

27. A tap which would fill a cistern in 3 hours, and a plug which would empty it in 7 hours, are both opened at the same instant, when the cistern is empty. How long will they take to fill the cistern?

28. A tap and a plug when both open fill an empty cistern in 120 minutes, whilst the tap alone would fill it in 40 minutes. How long would it take the plug to empty the full cistern?

29. Find graphically (to the nearest minute) the times when the hands of a clock are (1) coincident in direction, (2) at right angles, (3) 20 minute divisions apart between the times given below:

(i) 2 o'clock and 3 o'clock. (ii) 6 o'clock and 7 o'clock.

(iii) 9 o'clock and 10 o'clock. (iv) 11 o'clock and 12 o'clock.

30. If 4 men, or 5 women, or 6 boys can do the same amount of work in the same time, how long will one man, one woman, and one boy take to do a piece of work which 4 men do in $2\frac{1}{2}$ days?

31. A, B, and C together do one-quarter of a piece of work in one day, B and C do one-half in 4 days, C completes it in 6 days more. How long would A and B take to do it, working alone?

32. Two plugs are opened in a cistern containing 192 gallons of water: after 3 hours one of the plugs becomes stopped, and the cistern is emptied by the other in 11 more hours. Had 6 hours elapsed before the stoppage, it would have required only 6 hours more to have emptied the cistern. How many gallons will each hole discharge in an hour, supposing the discharge uniform?

33. Two workmen of unequal efficiency can make a drain in 60 days. After working for 20 days, A falls ill, and his place is taken by C, whose efficiency is the same as that of B, and they get the drain finished in 80 days. In how many days could A or B alone have made the drain?

34. A man mixes tea at 1*s.* 1*d.* per lb. with tea at 1*s.* 11*d.* per lb., and sells the mixture, without gain or loss, at 1*s.* 4*d.* per lb.: find the percentages of the two kinds in the mixture.

35. A man mixes tea at 1*s.* per lb. with some at 2*s.* 3*d.* per lb., and sells the mixture at 1*s.* 4*d.* per lb., without gain or loss. How many lbs. of each kind did he sell?

36. Two taps running together would fill a cistern in 15 hours. After running together for 6 hours, one is turned off, and the other fills the cistern in 30 hours more. In what time could each fill the cistern, running separately?

37. A tap which would fill a cistern in 4 hours is opened. After it has been running an hour, a second tap is opened which would, running alone, fill the cistern in 8 hours. At the end of a second hour, a plug is opened, and the cistern which was empty at the start, is empty again in 7 hours from the start. How long would the plug, running alone, take to empty the full cistern?

38. A cyclist rides 3 miles an hour faster downhill than uphill, and takes the same time to ride 22 miles downhill and 48 uphill that he takes to ride 50 miles downhill and 27 miles uphill. What are his speeds uphill and downhill?

39. The population of a town increases uniformly, and in each period of 3 years the increase is 20 per cent. of the population at the beginning of that period. If the population was 90,000 in January 1903, what will it be in January 1906, and 1912, and what was it in January 1897?

By drawing a graph, find the population approximately in January of each year from 1900 to 1905.

40. A and B are points 20 miles apart. At noon one man starts from A to walk to B at the rate of 4 miles an hour, and at 2 p.m. another man starts after him on a bicycle at 10 miles an hour. Draw a diagram on ruled paper to show how far they are apart at any given time, and at what times they pass any given point between A and B.

[Scale to be 5 miles = 1 inch, and 1 hour = 1 inch.]

Also find from the diagram when and where the cyclist overtakes the man walking.

261. *Trace the graph of* $25x^2 + 64y^2 = 1600$.

This may be written $\dfrac{x^2}{64} + \dfrac{y^2}{25} = 1.$

$$\frac{x^2}{64} = 1 - \frac{y^2}{25}; \quad \therefore x = \frac{\pm 8}{5}\sqrt{(25 - y^2)} \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

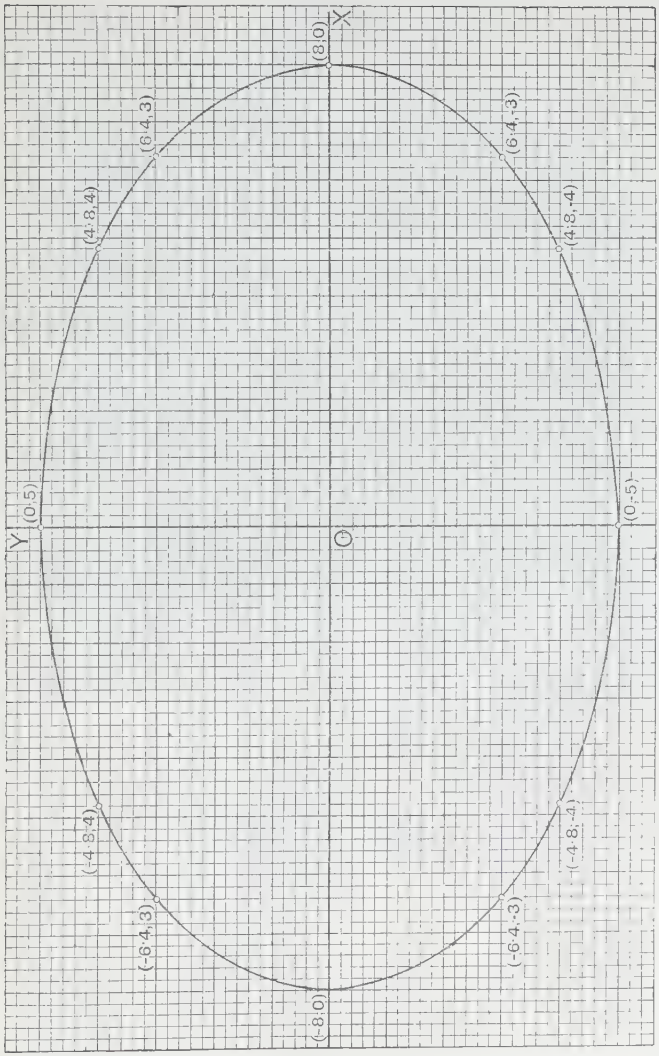
We see that for every value of y, we have two values of x, equal but of opposite sign;

$\therefore$ the curve is symmetrical about the axis of y.

In the same way it may be shown that the curve is symmetrical about the axis of x.

Also from equation (1) we see that y must lie between the values ± 5, or x would be imaginary.

By solving for y in the same way, we see that x must lie between the values ± 8.


X
Y
O
(8,0)
(6,4.3)
(6,-4.3)
(4,8.4)
(4,-8.4)
(0,5)
(0,-5)
(-4,8.4)
(-4,-8.4)
(-6,4.3)
(-6,-4.3)
(-8,0)

Particular Points.

When $y=0$, $x=\pm 8$.

..... $y=\pm 5$, $x=0$.

..... $y=\pm 4$, $x=\pm\frac{8\times 3}{5}=\pm 4{\cdot}8$.

..... $y=\pm 3$, $x=\pm\frac{8\times 4}{5}=\pm 6{\cdot}4$.

We can now draw the curve, and see that it is a closed curve of an oval shape. It is called an **Ellipse.**

262. In connection with the graph of any equation of the second degree, the following should be noted:

(1) **If the coefficients of x^2 and y^2 are equal and there is no term involving the product xy, the graph is a circle.**

(2) **If the terms of the second degree form a perfect square, the graph is an open one-branched curve, a parabola.**

(3) **If the terms of the second degree have no real factors, the graph is a closed oval curve, an ellipse.**

If the conditions (1) are also satisfied the curve is a circle.

(4) **If the terms of the second degree have real and different factors, the graph is a two-branched curve with asymptotes, a hyperbola.**

263. *Two taps running together fill a cistern in* 15 *hours. Running separately one tap takes* 16 *hours longer than the other to fill the cistern. Find how long each tap takes to fill the cistern.*

If x and y be the number of hours taken by the taps respectively to fill the cistern,

$$x-y=16,\quad \frac{1}{x}+\frac{1}{y}=\frac{1}{15},$$

$$\frac{xy}{x+y}=15;$$

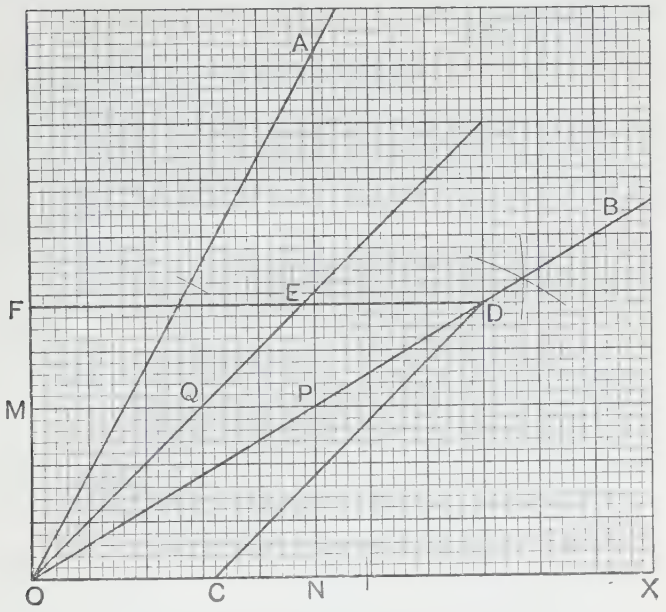
$$\therefore\quad \frac{xy}{x^2-y^2}=\frac{15}{16};$$

$$i.e.\quad \frac{\frac{2y}{x}}{1-\frac{y^2}{x^2}}=\frac{15}{8};$$

$$\therefore \text{ if } \tan\alpha = \frac{y}{x}, \quad \tan 2\alpha = \frac{15}{8}.$$

Hence the following construction.

Let $\frac{1}{10}$th of an inch represent one hour, measured horizontally.



Draw $\angle$ AOX whose tangent $= \frac{15}{8}$.

Bisect it by OB. Then $\frac{PN}{ON} = \frac{y}{x}$;

$\therefore$ making MQ = PN, OP, OQ are the graphs of the taps.

Take OC = 16, and draw CD to OQ to meet OP at D.

DE = 16 hours, $\therefore$ FE and FD are the reqd. times, 24 hours and 40 hours.

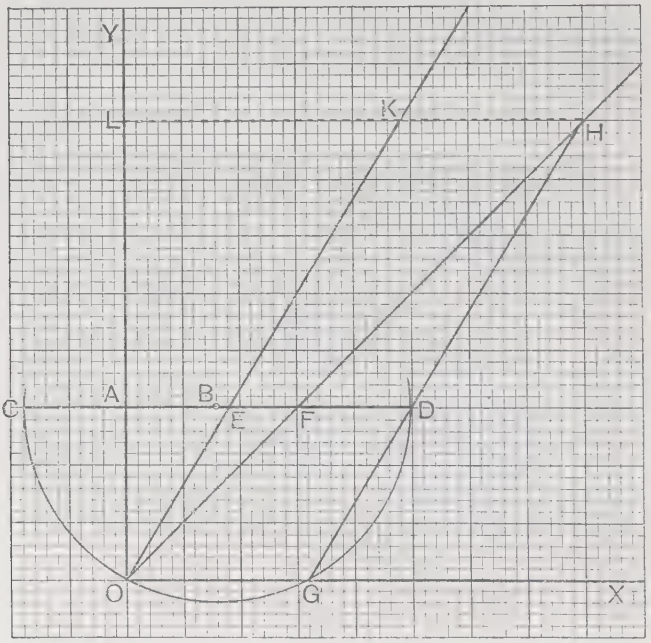
264. *Second Method.*

As in the first method $\frac{x^2 - y^2}{xy} = \frac{16}{15}$,

i.e. $\frac{x}{y} - \frac{y}{x} = \frac{16}{15}$.

We thus have to find two ratios $\left(\frac{x}{y}, \frac{y}{x}\right)$ such that their product is unity, and their difference $\frac{16}{15}$.

This amounts to finding two ratios $\left(\frac{15x}{y}, \frac{15y}{x}\right)$ whose product is 15^2 and difference 16.



Along OY take OA (1·5″) to represent 15, and AB (·8″) at rt. ∠s to it to represent 8 (half 16).

With centre B and rad. BO describe the circle COD and draw the diameter CBD.

$CA \cdot AD = OA^2 = 15^2$, and $AD - AC = 2 \cdot AB = 16$;

$$\therefore \quad AD = \frac{15x}{y} \text{ and } AC = \frac{15y}{x};$$

$$\therefore \quad \frac{AO}{AC} = \frac{15}{AC} = \frac{x}{y} = \text{ratio of the times taken by the two taps to fill the cistern.}$$

Hence making AE = AC and AF = AO, and joining OE, OF, OE and OF are the graphs of the taps.

Taking one-tenth of an inch to represent an hour, make OG = 16 and draw GH || to OE to meet OF in H.

Draw HKL || to OX. HK = 16;

∴ LK and LH represent the reqd. times; 24 and 40 hours.

Examples. XL. b.

The following problems should be solved graphically. They may be verified algebraically.

1. A and B are at two places P and Q 30 miles apart. They start at the same instant to travel from P to Q and Q to P respectively and meet at R 12 miles from P. If A arrives at Q $2\frac{1}{2}$ hours after B arrives at P, find, graphically, their rates of travelling.

2. A tap A fills one-half of a cistern in the same time that a tap B fills three-quarters of an equal cistern. Starting at the same time B fills the one cistern 2 hours sooner than A fills the other. Find how long each tap takes to fill its cistern.

3. Two men, starting at the same time from two places A and B, travel from A to B and B to A respectively at uniform rates. They are 30 miles apart in 3 hours and meet in 9 hours at a point 30 miles from A. How far is it from A to B, and what are their rates of travelling?

4. Two men, walking from A to B and B to A respectively, and starting at the same time, meet in 7 hours at a point $26\frac{1}{2}$ miles from A, and are 15 miles apart in 4 hours. How far is it from A to B, and what are their rates of travelling?

5. A man, who can row 2 miles against a stream in the same time as 3 miles with it, rows 15 miles and back in $6\frac{1}{4}$ hours. Find his times of rowing up and down.

6. Two pipes together fill a cistern in $2\frac{6}{7}$ hours. Running singly, one takes 6 hours longer than the other to fill it. How long does each tap take to fill it?

7. A man and a boy take $3\frac{1}{3}$ days to do a piece of work. Working singly the boy would take 12 days longer than the man to do it. How long does each take to do it?

8. A and B together do a piece of work in $9\frac{3}{8}$ days. Working singly A takes 10 days longer than B to do the work: how long does each man take?

9. A and B start together from the foot of a mountain to go to the summit. A would reach the summit half-an-hour before B, but missing his way, goes a mile and back needlessly, during which he walks at twice his former pace. He does the rest of the walk at his original pace, and reaches the summit 6 minutes before B. C starting 20 minutes after A and B, and walking at the rate of $2\frac{1}{7}$ miles per hour, reaches the summit 10 minutes after B. Find the rates of walking of A and B, and the distance from the foot to the summit of the mountain.

10. A steamer does 6 miles up stream and down again in 2 hours and 40 minutes, and the current runs at the rate of 3 miles an hour. What is the steamer's rate in still water?

Plot out the graphs of :

11. $64x^2+y^2=64$.	12. $x^2+25y^2=25$.	13. $16x^2+9y^2=144$.
14. $64x^2-y^2=64$.	15. $x^2-y^2=25$.	16. $16x^2-9y^2=144$.
17. $x^2-y^2=64$.	18. $9y^2-x^2=0$.	19. $xy=0$.
20. $x^2+y^2=0$.	21. $(x-5)(y-6)=0$.	22. $(x-5)^2+(y-6)^2=0$.
23. $y=\frac{1}{x-1}$.	24. $9x^2+16y^2=144$.	25. $y=(x-1)(x-2)$.

Examples. XL. c.

MISCELLANEOUS PROBLEMS.

1. A and B run a race ; B has 50 yards start, but A runs 20 yards while B runs 19 ; what must be the length of the course, that A may come in a yard ahead of B ?

2. A man buys a certain quantity of apples to divide among his five children in succession. To the eldest he gives half the whole, all but 8 apples ; to the second he gives half the remainder, all but 8 apples ; to the third half the second remainder, all but 8 apples ; to the fourth half the third remainder all but 8 apples. To the fifth he gives the 20 apples which remain. Find how many he bought.

3. One army contains 3 men for every 2 in another army. In a battle the first army loses 1600 men and the second 600 men. After this there are only 4 men in the first army for every 3 in the other. How many men were there in each army at first ?

4. A trader begins business with a certain capital. The first year he spends £100 on himself, and at the end of the year finds that the rest of his capital has increased by one-third. At the end of the second year, after again spending £100 on himself, he finds that the capital with which he began the second year has (in a similar manner) increased by one-third, and that he now has £1466. 13*s*. 4*d*. How much had he to begin with ?

5. Find two integers whose sum shall be 21, and which shall be such that the sum of the digits of one of the integers shall be double the other integer.

6. Find what quantity of tea at 2*s*. per lb. must be mixed with 30 lb. of tea at 3*s*. per lb. in order that 2*s*. may be gained by selling 10 lbs. of the mixture at 2*s*. 6*d*. per lb.

7. A man sets out to walk from A to B, a distance of 16 miles, at the rate of $3\frac{1}{2}$ miles an hour. Three-quarters of an hour after, a man sets out from B to meet him, at the rate of 4 miles an hour. When and where will they meet ?

8. A and B possess together £22. After A has given to B one-fourth of his money, then one-third of what remained to him, and then £4 more, he has left only one-tenth of what B now has. How much money had each originally ?

9. In a bicycle race A goes x feet per second and B goes $\frac{7x}{11}$ miles per hour. After 3 minutes A is 88 yards ahead of B : find x.

10. There is a number from which two other numbers are formed, one by adding 4 to it and the other by subtracting 4 from it ; and the difference of the squares of these numbers is 144. What is the number ?

11. A dealer buys $2a+b$ sheep at c shillings each, and sells $a+b$ of them at a profit of d shillings each, disposing of the remainder at a loss of d shillings each: find his gain by the whole transaction.

12. Of a swarm of bees clustered on a tree, the square root of half their number flew away. Eight-ninths of the original number then departed, leaving but two behind. How many were there at first?

13. A town is supplied with water from a reservoir into which the water has first to be pumped. The pumps if worked constantly could, in the case of no leakage or consumption of water, fill the reservoir in x days. They, however, work only on week-days, and for only 12 hours a day. Also the town consumes a reservoir-full in $3x$ days, using it night and day, and all days, and by leakage a reservoir-full is lost in $39x$ days. On Monday morning when the pumping begins the reservoir is empty. Find x, that on the next Saturday evening but two, when the pumping leaves off, the reservoir may be for the first time full.

14. A clock is set going at 12 o'clock, but loses four minutes an hour; at what true time between 2 and 3 o'clock will its hands point in opposite directions?

15. A grocer buys a number of eggs at 6*s.* 6*d.* per hundred. He sells all but 69 of them at the rate of 11 for a shilling, and finds that he has received 30*s.* more than he gave for the whole number. How many eggs did he buy?

16. There are two mixtures of wine and water, one contains twice as much water as wine, and the other three times as much wine as water. How much must be taken from each mixture to fill a pint vessel in which the wine and water are equally mixed?

17. A started six minutes before B and walked nine times round a circular path, finishing just as B completed his seventh round. A now rested for five minutes, while B went on with speed increased to half as much again as before, and just as he completed his twelfth round A, who had resumed his walk with double speed, finished his sixteenth round. How long do A and B require at first to walk round a path?

18. A litre of water weighs a kilogram, and a litre of another liquid weighs 1·34 kilograms. A mixture of the two weighs 1·27 kilograms per litre. Determine the volume of each in a litre of the mixture.

19. Potatoes are sold so as to gain 25 per cent. at 6 lbs. for 5*d.*: find the gain per cent. when they are sold at 5 lbs. for 6*d.*

20. A grocer gains 20 per cent. by selling at 2*s.* a lb. a mixture formed by mixing with 7 lbs. of a common tea 2 lbs. of a better kind. But if he had mixed 7 lbs. of the latter with 2 lbs. of the former kind, he would have lost 20 per cent. by selling the mixture at that price. What did each kind of tea cost him per lb.?

21. A man has three sons, born at equal intervals of time, whose united ages are at present equal to his own age. Show that when the age of the middle one is double of what it is now he will be exactly half his father's age.

22. A cricketer had a certain number of innings during the season: he finds that if he had not played the last, when he only made 5 runs, or if he had played an extra innings and made 31 runs, he would in either case have increased his average by 1. How many innings did he have, and what was his average?

23. Two purses contain money in such proportion that for every £1 there is in one of them there is 15*s.* 6*d.* in the other; and if £1 be taken out of the latter and put into the former, the difference of the contents will then be £3. 2*s.* 6*d.*; how much money is there in each purse?

24. I take a party by rail. Three times the number of those in my party who require whole tickets is equal to twice the number of those who require half-tickets. Find the number in my party, if 14 tickets suffice for all.

25. In a division in parliament, the Government had a majority of 18, and it was found that 53 per cent. of the whole number had voted for the Government. Find how many voted on each side.

26. The age of a father is twice the sum of the ages of his two sons, the elder of whom is twice as old as the younger. Nine years hence the father will be three times as old as his younger son. Find their ages.

27. I spend half of a certain sum of money in buying wheat at 28*s.* a quarter, and the other half in buying barley at 24*s.* a quarter. If the wheat had cost 32*s.* 3*d.*, and the barley 26*s.* 8*d.* a quarter, I should have altogether spent £13. 5*s.* more. How many quarters of each did I buy?

28. On a certain journey I walked at $3\frac{1}{2}$ miles an hour, until the number of miles left to be walked was three less than the number of hours I had been walking. Then I walked the rest of the way at 4 miles an hour. If the whole journey occupied 10 hours 55 minutes, how many miles long was it?

CHAPTER XLI.

PAPERS FOR REVISION.

XLI. a.

1. Write down the square of $x+x^{-1}-1$, and the cube of $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

2. Find the value to 2 decimal places of $\dfrac{15+\sqrt{10}}{15-\sqrt{10}}+\dfrac{30-\sqrt{10}}{15+\sqrt{10}}$.

3. If $a:b=c:d$, then $a+\dfrac{1}{b}:b+\dfrac{1}{a}=c+\dfrac{1}{d}:d+\dfrac{1}{c}$.

4. If $\mathsf{A}\propto\mathsf{BC}$, and $\mathsf{B}^2\propto\mathsf{AC}$, prove that $\mathsf{A}\propto\mathsf{B}^{\frac{3}{2}}\propto\mathsf{C}^3$.

5. A can run round a circular path 5 times while B can run round another 6 times. If their rates are as 9 to 7, compare the lengths of the paths and determine how much per cent. A must increase his speed to run round his path 6 times while B with unaltered speed runs round his 5 times.

6. Find by logarithms the cube root of 4913.

7. Solve $x^2-3\sqrt{2x^2-x-2}=x(1-x)$.

XLI. b.

1. If $a:b=c:d$, prove $\dfrac{a^2}{b^2}=\dfrac{2a+c}{2b+d}\times\dfrac{2a-c}{2b-d}$.

2. Find the cube root of 3 to 3 places.

3. The distance fallen by a body varies as the square of the time occupied, and it falls 16 feet in the 1st second. Find how far it falls in 6 seconds, and how far in the 6th second.

4. Prove that ax^3+bx^2+cx+d is divisible by $x-1$ if $a+b+c+d=0$.

5. Simplify $2\sqrt{72}+7\sqrt{8}-\sqrt{98}+\dfrac{6}{\sqrt{2}}$.

6. Find $\log_{10}525$ and $\log_{10}4{\cdot}41$, given $\log 2$, $\log 3$, $\log 7$.

7. How much are eggs a score when a rise of 20 per cent. in the price would make a difference of 80 in the number for a sovereign?

XLI. c.

1. Solve $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=3$.

2. Find the sides of a right-angled $\angle$ whose area is 30 sq. in. and hypotenuse 13 in.

3. Divide 35 by $2\sqrt{6}-\sqrt{3}$.

4. Simplify $\left(8a^{\frac{1}{2}}-27x^{-\frac{1}{2}}\right)\div\left(2a^{\frac{1}{6}}-3x^{-\frac{1}{6}}\right)$.

5. If $a:b=c:d$, then $\dfrac{a^{2n}+b^{2n}+c^{2n}+d^{2n}}{a^{-2n}+b^{-2n}+c^{-2n}+d^{-2n}}=(abcd)^n$.

6. In which is the greater waste, in cutting a square out of a circle or a circle out of a square? Find the ratio of the waste to the original area in each case.

7. Given that the area of a circle varies as the square of the radius, prove that the area of a circle of radius 13 inches = the sum of the areas of two circles of radius 5 and 12 inches.

XLI. d.

1. Express $(a^2+b^2)(c^2+d^2)$ as the sum of two squares.

2. Solve $x^2-6x+16=4\sqrt{x^2-6x+12}$.

3. Draw the graph of $y=10^x$ from $x=-1$ to $x=1{\cdot}5$, and find from it the common logarithms of 1·25, 23, and 6·3.

4. There are two vessels A and B each containing a mixture of water and wine, A in the ratio of 2 to 3, B in the ratio of 3 to 7. What quantities must be taken from them to form a third mixture which shall contain 5 gallons of water and 11 of wine?

5. Show graphically that the ratio $a:b$ lies in value between the ratio $na:nb+p$ and the ratio $na:nb-p$.

6. Simplify $2^{-1}\cdot 3^{\frac{1}{2}}\cdot 6^{\frac{3}{2}}+9^{-\frac{1}{3}}\cdot 12^{\frac{2}{3}}\cdot 2^{\frac{1}{6}}+2^{-\frac{3}{2}}$.

7. The volume of a cube is 17·5 cubic feet; find by logarithms the length of its edge.

XLI. e.

1. Simplify $\sqrt{\dfrac{22+6\sqrt{5}}{8-3\sqrt{5}}}$ and find its value to 3 places.

2. Factorize $a^3+b^3+c^3-3abc$, and divide $x+y+z-3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}$.

3. Solve $\sqrt{2x+8}-2\sqrt{x+5}+2=0$.

4. If $a:b=c:d$, show by similar triangles that $\dfrac{a^2+b^2}{b^2}=\dfrac{c^2+d^2}{d^2}$.

5. The area of a right-angled $\triangle$ is 84 sq. in., and its perimeter 56 in. Find the sides.

6. The volume of a sphere varies as the cube of its radius. Prove that three spheres of radii 3, 4, 5 are together equal in volume to one of radius 6.

7. If $3^x=9\times 3^{\frac{1}{x}}$, find x to 2 decimal places.

XLI. f.

1. The third proportional to two numbers is 162, and the mean proportional 6. Find the numbers.

2. A man receives a fixed allowance of £75 a year, but increases his expenditure by 25 per cent. every year. At the end of 4 years he is £69 in debt. How much did he save the first year? *Represent this graphically.*

3. Which is the greater, 3 : 4 or $3x^2+6x+1 : 4x^2+8x+1$?

4. If $\dfrac{a}{b}=\dfrac{c}{d}$, $a:d=(a+b)(a-c):(b-d)(c+d)$.

5. Prove $\log_a N=\log_a b\,.\,\log_b N$; and find $\log\sqrt{5}$ to the base ·008.

6. Three diamond rings weigh 141, 116, 79 grains, and it is known that the weights of the gems are 3, 2, 1 carats. The 1st ring is worth £82, the 2nd £36. 16s. What is the value of the 3rd? [1 carat = 4 grains, and the value of a diamond varies as the square of the weight.]

7. Find the value of $\sqrt{1\cdot5+\sqrt{2}}$ graphically and algebraically.

XLI. g.

1. Find a number which added to each of the numbers 6, 8, 10, 13 will make them proportionals.

2. Find the value of $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$ when $x=\dfrac{2a}{1+a^2}$.

3. Find the number whose logarithm is $\frac{1}{3}$ of $\bar{2}\cdot8471$.

4. If a, b, c are in continued proportion, $a+2b+c=\dfrac{(b+c)^2}{c}$, and

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{a^3+b^3+c^3}{a^2b^2c^2}.$$

5. Given that $w\propto x+y$ and $y\propto z^2$, and when $z=2$, $w=26$, $x=1$, $y=12$; express w in terms of x and z.

6. Solve (i) $\left.\begin{aligned} x+\sqrt{x+y}&=12-y \\ x^2+y^2&=41 \end{aligned}\right\}$,

(ii) $3^{2x}-2\times 3^{x+1}=567$.

7. A walks from Cheltenham to Gloucester at 4 miles an hour: B starts at the same time from Gloucester and bicycles to Cheltenham and back at 10 miles an hour. Find *graphically* where B passes A, the distance between the two places being 8 miles.

XLI. h.

1. Express $(a^2+b^2)(c^2+d^2)(e^2+f^2)$ as the sum of two squares.

2. The height of a spiral staircase is 40 feet and the internal diameter 3·183 feet. The inner hand-rail makes 3 complete revolutions: find its length.

3. Solve $2x^3+7x^2+7x+2=0$.

4. Prove from the definition that $\log_y \frac{x}{yz}=\log_y x-1-\log_y z$.

5. If a, b, c, d are in continued proportion $(b+c)(b+d)=(a+c)(c+d)$, and $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

6. Find *geometrically* the value of $\sqrt{1+\sqrt{5}}$.

7. Find $\log(2\sqrt{7}\div 3\sqrt{11})^{\frac{1}{6}}$.

XLI. i.

1. Add together $\sqrt{20}$, $3\sqrt{125}$, $4\left(\frac{80}{9}\right)^{-\frac{1}{2}}$, $\frac{9}{\sqrt{5}}$, $\sqrt{\frac{324}{5}}$ and $\sqrt[4]{\frac{25}{16}}$.

2. Find the square root of

$$a^{-4}-6a^{-3}b+\frac{29}{3}a^{-2}b^2-2a^{-1}b^3+\frac{1}{9}b^4.$$

3. Given $\log 2=\cdot 3010$, $\log 3=\cdot 4771$, find $\log 5$, $\log 1\cdot 2$ and $\log \cdot 00675$.

4. If $y\propto\frac{1}{x}$ and $x=2$ when $y=2$ draw the graph of the resulting equation, and thus find the value of y when $x=3\cdot 14$.

5. Two cisterns connected by a pipe contain 25 and 24½ gallons of water. How much must flow from the first into the other that their contents may be in the ratio of 5 to 6?

6. If a, b, c, d are proportional,

$$(a-b)(c-d)=\frac{b}{d}(c-d)^2=\frac{d}{b}(a-b)^2.$$

7. Solve (i) $x^2+2\sqrt{x^2+2x+3}=12-2x$,

(ii) $\frac{1}{x+\sqrt{2-x^2}}+\frac{1}{x-\sqrt{2-x^2}}=\frac{x}{2}$.

XLI. k.

1. Draw on squared paper the graph of $\sqrt{\frac{x}{3}}+\sqrt{\frac{y}{2}}=1$. Draw also

$$\frac{x}{3}+\frac{y}{2}=1.$$

Find, by counting squares, the area of the triangle whose vertices are (0, 0), (3, 0), (0, 2), and the ratio of the parts into which the curve divides it.

2. Solve $\sqrt{x}+\sqrt{a+x}=\dfrac{2x}{\sqrt{a+x}}$.

3. A and B start to run a race to a certain post and back again. A returning meets B at 80 yards from the post and arrives at the starting point 1 minute before him. If he had then returned immediately to meet B, he would have met him when B had still to run $\frac{1}{5}$th of the distance between the post and the starting-place. Find the length of the course and the duration of the race.

4. α, β are the roots of $x^2+px+q=0$. Find $\alpha\beta^{-1}+\alpha^{-1}\beta$.

5. Extract the square root of $16+\sqrt{252}$.

6. Find the product of $a^{\frac{1}{3}}$, $a^{-\frac{3}{4}}$, $\sqrt[3]{a^4}$, $a^{1\frac{1}{2}}$, $\sqrt[8]{a^{\frac{25}{3}}}$ and $\left(a^{-\frac{7}{4}}\right)^{\frac{7}{6}}$.

7. If x men do as much work as y women, and p women as much as q boys, what should be the ratio of a man's wages to a boy's?

XLI. l.

1. A man loses $\frac{1}{3}$ of his money; then wins £10; loses $\frac{1}{5}$ of what he then has, and wins £20, and finds that he has exactly what he had at starting. What had he?

2. Simplify $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

3. If $x \propto \dfrac{z}{y^2}$, and $z^2 \propto \dfrac{y}{x}$, prove that $x \propto \dfrac{1}{y} \propto \dfrac{1}{z}$.

4. To do a piece of work A takes m times as long as B and C together, B n times as long as C and A together, C p times as long as A and B together. Prove $\dfrac{1}{m+1}+\dfrac{1}{n+1}+\dfrac{1}{p+1}=1$.

5. From the equations $x^2+y^2=25$, $xy=12$ find the ratio $x:y$.

6. Verify this by drawing the graphs of the two equations and finding the equations of the lines joining the origin to their intersections.

7. Find by logarithms the 4th root of $\dfrac{3546}{121}$ and verify by extracting the square root of the square root.

XLI. m.

1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\sqrt{ab}+\sqrt{cd}+\sqrt{ef}=\sqrt{(a+c+e)(b+d+f)}$.

2. Simplify $\frac{\sqrt{5}+1}{\sqrt{5}-\sqrt{2}}+\frac{\sqrt{5}-1}{\sqrt{5}+\sqrt{2}}-\frac{2(\sqrt{5}-\sqrt{2})}{\sqrt{89-28\sqrt{10}}}$.

3. There are two vessels, one containing 7 gallons of water and 3 of spirit, the other 4 gallons of water and 6 of spirit. How much must be taken from each to fill a 10 gallon vessel with equal parts of water and spirit?

4. Find the mean proportional between $4{\cdot}\dot{6}$ and $10{\cdot}5$.

5. Given $\log \sqrt[3]{275}=\cdot 8131$ and $\log 22=1{\cdot}3422$, find $\log 2$.

6. In the equation $ax^2+bx+c=0$, $2b^2=9ac$, find the ratio of the roots.

7. The value of a silver coin varies as the product of the thickness and the square of the diameter. Two silver coins have their diameters in the ratio 5 : 4. Find the ratio of their thicknesses if the value of the 1st is twice that of the 2nd.

XLI. n.

1. By dividing a line in extreme and mean ratio, find by measurement $\frac{1}{2}(\sqrt{5}-1)$.

2. If $a \propto b$ and c jointly, and $b \propto d^2$, and $c \propto$ the reciprocal of a, prove that $a \propto d$.

3. If $\log 75$ lie between 2 and 3, what integer is the base?

4. How many figures are there in 5^{100}, given $\log 2=\cdot 30103$?

5. Solve $\sqrt{3x-5}+2\sqrt{x-2}=\sqrt{5x+1}$.

6. A father gave his son a certain sum of money, telling him that at the end of every year he would give him as much as he then had left. The son spent £100 a year, and at the end of 4 years had nothing left. How much did he receive at first?

7. A and B start to run a race. When A has run c yards, B is d yards behind. B then increases his pace in the ratio $m : n$. The race lasts t minutes longer and results in a dead heat. Find the distance and the rates of running.

XLI. p.

1. Simplify the ratio $\frac{(ax+by)^2-(a-b)(x+z)(ax+by)+(a-b)^2xz}{(ax-by)^2-(a+b)(x+z)(ax-by)+(a+b)^2xz}$.

2. The perimeter of a right angled triangle is 6 times the least side. Find the ratio of the sides containing the right angle.

3. Find $\log \sqrt[3]{(\frac{5}{12})}$.

4. If $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}$, prove that $x^x y^y z^z=1$.

5. The square of the time during which a body will slide down a smooth incline varies as the square of the length and inversely as the height. If the time is 1 second when the height is 4 feet and length 8 feet, what must be the height of a plane 3 feet long that a body may slide down in half a second?

6. In a walking race A gives B a start of two-thirds of a mile. After A has walked for 36 minutes he overtakes B, but 18 minutes later he meets with an accident which detains him 27 minutes. On resuming the race, A walks for an hour and a half and again overtakes B two-thirds of a mile from the winning post. Find the length of the course.

7. If $a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz)$, then

$$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}.$$

XLI. q.

1. Find the square root of $\frac{17}{7}\frac{8}{9}-\frac{20}{7}ab^{-1}+\frac{9}{16}a^{-2}b^2-\frac{15}{2}a^{-1}b+\frac{4}{49}a^2b^{-2}$.

2. Put into a proportion the statement

$$(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d),$$

and prove that $a:b=c:d$.

3. Two travellers start towards each other from two places, one walking a mile an hour faster than the other. They meet a mile from the middle of the distance, and find that the time of the faster over the whole distance would be $3\frac{1}{2}$ hours. Find the distance.

4. The weight of a spherical shell is $\frac{7}{8}$ of what it would be if it were solid. Compare the inner and outer radii, given that the volume varies as the cube of the radius.

5. In a proportion the sum of the means = 8, sum of extremes = 16, the sum of squares of all the terms = 260. Find the proportion.

6. Express in decimals 10^{-1}, 10^{-5}, 10^{-25}, 10^{25}, 10^{5}, 10^{75}. Draw the graph of $y=10^x$ on a large scale for values of x between -1 and 1.

Find from the graph the values of 10^{2}, 10^{3}, 10^{-75}.

Check the results by means of a table of logarithms.

7. Find $3^{\frac{1}{8}}$ by extracting square roots, and state to how many places your result is correct.

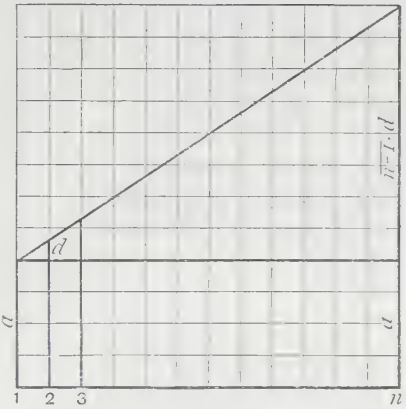
CHAPTER XLII.

ARITHMETICAL PROGRESSION.

265. A series whose terms proceed by a common difference is called an **Arithmetical Progression.**

The common difference = 2nd term − 1st = 3rd − 2nd = etc., 3, 5, 8 ... form an A.P., and so do 20, 17, 14.

266. *To find the nth term of an* A.P.



Let $a =$ 1st term, $d =$ common difference.

2nd term $= a + d$.

3rd term $= a + 2d$, etc.

$\therefore$ **nth term** $= \mathbf{a + (n-1)d}$.

267. *To find the sum of n terms of an* A.P.

Let $a =$ 1st term, $d =$ common difference, $l =$ last term,

$s =$ sum of n terms.

$s = a + (a+d) + (a+2d) + \ldots + l$.

$s =$ the sum of the same terms in reverse order

$= l + (l-d) + (l-2d) + \ldots + a$.

By addition

$$2s = (a+l) + (a+l) + (a+l) + \ldots = n(a+l).$$

$$\therefore \quad \mathbf{s = \frac{n}{2}(a+l)} \qquad \ldots\ldots(1)$$

But

$$l = a + (n-1)d \qquad \ldots\ldots(2)$$

$$\therefore \quad s = \frac{n}{2}\{a + a + (n-1)d\}$$

$$= \mathbf{\frac{n}{2}\{2a + (n-1)d\}} \qquad \ldots\ldots(3)$$

268. *To insert n Arithmetic Means between x and y, i.e.* to form an A.P. of $n+2$ terms beginning with x and ending with y.

$$y = \text{the } (n+2)\text{th term} = x + (n+1)d.$$

$$\therefore\ d = \frac{y-x}{n+1}.$$

$$\therefore \text{ the means are } x + \frac{y-x}{n+1},\quad x + \frac{2(y-x)}{n+1},\quad x + \frac{3(y-x)}{n+1}, \text{ etc.}$$

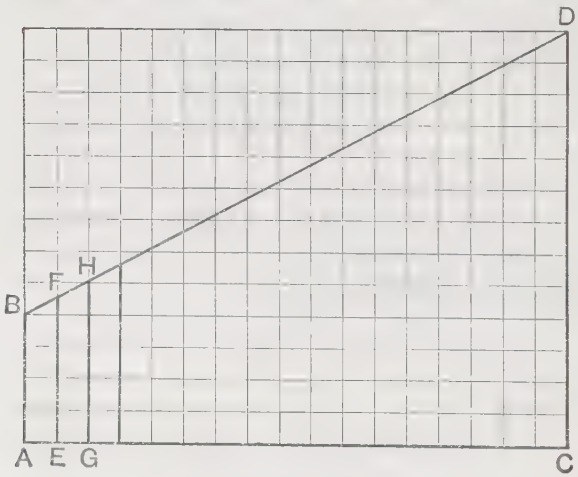
269. *To find the Arithmetic Mean between x and y, i.e.* to insert one mean between x and y.

Let A be the required mean.

$$\therefore\ A - x = y - A.$$

$$\therefore\ A = \frac{x+y}{2}.$$

270. *Graphic proof of the formula for the sum of an* A.P.



Let AB, CD represent the 1st and last terms a, l of an A.P., the number of terms being n.

Let EF, GH, etc., represent the 2nd, 3rd, etc. Let AE, EG, etc., each = 1. There are evidently $n-1$ of these intervals.

Now the area of a trapezium = half the sum of the parallel sides × the distance between them.

$\therefore \frac{1}{2} \cdot 1 \cdot \{(1\text{st ordinate} + 2\text{nd}) + (2\text{nd} + 3\text{rd}) + \ldots + (\overline{n-1}\text{th} + n\text{th})\}$

$=$ sum of trapeziums AF, EH, etc.

Here we have the sum of the series of terms except that we have only half the first and last.

$$\therefore S - \tfrac{1}{2}(a+l) = \text{the whole trapezium ABDC}$$
$$= \tfrac{1}{2}\,AC(AB + CD)$$
$$= \frac{n-1}{2}(a+l).$$

$$\therefore S = \frac{n-1}{2}(a+l) + \tfrac{1}{2}(a+l) = \frac{n}{2}(a+l).$$

Example 1. The 5th term of an A.P. is -5, and the 11th term -23: find the 30th term and the sum of 30 terms.

Let the 1st term be a, the common difference d.

$$a + 4d = 5\text{th term} = -5,$$
$$a + 10d = 11\text{th term} = -23;$$
$$\therefore 6d = -18; \quad \therefore d = -3, \ a = 7;$$
$$\therefore 30\text{th term} = 7 - 29 \times 3 = 7 - 87 = -80.$$

The sum $= \frac{n}{2}\{a+l\} = \frac{30}{2}\{7 - 80\} = -15 \times 73 = -1095.$

We might find the sum of 30 terms without finding the 30th term.

For $$s = \frac{n}{2}\{2a + (n-1)d\} = \frac{30}{2}\{14 - 87\} = -1095.$$

Example 2. Insert 5 arithmetic means between 11 and 37.

Here 11 and 37 with the intermediate terms form an A.P. of 7 terms.

The first term is 11, the 7th is 37.

i.e. $$11 + 6d = 7\text{th term} = 37;$$
$$\therefore 6d = 26; \quad \therefore d = 4\tfrac{1}{3}.$$

The required means are $15\frac{1}{3}$, $19\frac{2}{3}$, 24, $28\frac{1}{3}$, $32\frac{2}{3}$.

Example 3. How many terms of the series 15, 13, 11, ... make 55?

Here $s = 55$, $a = 15$, $d = 2\text{nd} - 1\text{st} = -2$; n is required.

Now $$s = \frac{n}{2}\{2a + (n-1)d\};$$

$$\therefore 55 = \frac{n}{2}\{30 - 2(n-1)\} = \frac{n}{2}\{32 - 2n\} = 16n - n^2.$$
$$n^2 - 16n = -55;$$
$$\therefore n = 5 \text{ or } 11.$$

It will be found on trial that the sum of 11 terms = the sum of 5 terms = 55.

Example 4. The sum of 5 terms in A.P. is 35: find the middle term.

To express 5 terms in A.P. so that their sum shall be as simple as possible we may write them down as $a - 2d$, $a - d$, a, $a + d$, $a + 2d$.

In this case $$5a = \text{the sum} = 35;$$
$$\therefore \text{the middle term } a = 7.$$

Examples. XLII. a.

1. Write down the 5th term of the series whose nth term is $n+2$.
2. ..... 13th ..... $3n+1$.
3. ..... 12th ..... $2n-1$.
4. ..... 25th ..... $3n-5$.
5. ..... 4th ..... $7n-4$.
6. ..... 1st ..... $9n-5$.
7. ..... 11th ..... $\frac{3n-5}{4}$.
8. ..... 7th .....$(n+1)$th.... $n+2$.
9. ..... 10th ..... $3n-1$.
10. ..... 14th ..... $4n-7$.
11. ..... 7th .....$(n+2)$th.... $n+3$.
12. ..... 9th ..... $3n-1$.
13. ..... 1st ..... $4n-3$.
14. Find the 12th term of $3+7+11+$ etc.
15. ,, 21st term of $6+4+2+$ etc.
16. ,, 13th term of $\frac{1}{2}+\frac{3}{4}+1+$ etc.
17. ,, 101st term of $20+23+26+$ etc.
18. ,, 17th term of $100+91+82+$ etc.
19. ,, nth term of $1+3+5+$ etc.

Find the last term and the sum of the following series:

20. $1+2\frac{1}{4}+3\frac{1}{2}+$ etc., to 12 terms.
21. $1+1\frac{3}{4}+2\frac{1}{2}+$ etc., to 12 terms.
22. $32+29+26+$ etc., to 50 terms.
23. $\cdot5+\cdot75+1+$ etc., to 20 terms.
24. $\cdot7+\cdot6+\cdot5+$ etc., to 31 terms.

Sum the series

25. $13+12\frac{1}{3}+11\frac{2}{3}+$ etc., to 40 terms.
26. $17+\frac{49}{3}+\frac{47}{3}+$ etc., to 51 terms.
27. $-7-5\frac{2}{3}-4\frac{1}{3}-$ etc., to 21 terms.
28. $1\cdot2+2\cdot4+3\cdot6+$ etc., to 16 terms.
29. $(3a-2b)+(4a-5b)+(5a-8b)+$ etc., to $2n$ terms.
30. $(x+1)+(x+3)+(x+5)+$ etc., to 15 terms.

Examples. XLII. b.

Find the last term and the sum of

1. $3+9+15+\ldots$ to 13 terms.
2. $2+7+12+\ldots$ to 11 terms.
3. $21+18+15+\ldots$ to 17 terms.
4. $2-5-12-\ldots$ to 12 terms.

Find the sum of

5. $1+8+15+\ldots$ to 40 terms.
6. $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\ldots$ to 9 terms.
7. $\frac{1}{2}+\frac{7}{10}+\frac{9}{10}+\ldots$ to 20 terms.
8. $1+3+5+\ldots$ to 80 terms.
9. $2\frac{5}{12}+2+2\frac{3}{12}+\ldots$ to 14 terms.
10. $\frac{1}{2}-\frac{2}{3}-1\frac{1}{6}-\ldots$ to 18 terms.

11. There are 40 stones in a row, 1 yard apart. How far does a boy travel in bringing them together one by one at the 1st stone (1) if he begins at the 1st, (2) if he begins at the last?

12. A travels 2 miles the 1st hour, $2\frac{1}{4}$ the 2nd, $2\frac{1}{2}$ the 3rd, and so on. B travels 4 miles an hour. When and where does A overtake B if they start together?

13. How many strokes does a clock make in 12 hours, if it strikes 1 for the half hours?

14. Sum to 10 terms the series whose nth term is $3n+4$.

15. A man receives 5 shillings the 1st week and $3d.$ more each week than the preceding. What does he get in 20 weeks?

16. How may 5 numbers in A.P. be expressed so as to make their sum as simple as possible? How may 4 be expressed?

Examples. XLII. c.

1. The 7th term of an A.P. is 20, and the 13th is 38; find the series.

2. ... 11th ..... 36, ... 20th .. 27; .....

3. ... 10th ..... 5, ... 17th .. 54; .....

4. ... 12th ..... 10, ... 20th .. 8; .....

5. Sum to n terms the series whose nth term is $4+5n$.

6. ..... $3+7n$.

7. The 5th term of an A.P. is 13, and the 9th is 25. What is the 7th term?

8. The 1st term of an A.P. is 3 and the 20th is 136. Find the common difference and the sum of the first 20 terms.

9. Insert 6 arithmetic means between 11 and 53.

10. ..... 8 ..... 5 and 11.

11. ..... 20 ..... 35 and -28.

12. ..... 5 ..... $x+y$ and $x-y$.

13. Find the arithmetic mean between 17 and 33.

14. ..... $\frac{1}{2}$ and $\frac{1}{3}$.

15. Find the sum of 30 consecutive odd numbers of which the last is 127.

16. The 7th term of an A.P. is 30 and the 13th is 42. Find the 1st term and the sum of 12 terms.

17. Find the sum of all the even numbers from 2 to 38 inclusive.

18. The sum of the 8th and 4th terms of an A.P. is 24, and the sum of the 15th and 19th is 68. What is the series?

19. The sum of the first seven terms of an A.P. is 140, and the product of the 1st and 7th is 256. Find the terms.

20. A workman is to be paid 1*s.* for his 1st day's work, 1*s.* 1*d.* for the 2nd day, 1*s.* 2*d.* for the 3rd, and so on. Find how much more he earns in the 2nd week than in the 1st, if he works 6 days in the week.

21. A and B go round the world, of which the circuit is 25668 miles. A goes east 1 mile the 1st day, 2 the 2nd, 3 the 3rd and so on. B goes west uniformly at 20 miles a day. When and where do they meet?

22. Insert 5 arithmetic means between $a-2b$ and $3a+b$.

23. The last term of an A.P. is 10 times the 1st, and the last but one = the sum of the 4th and 5th. Find the number of terms and show that the common difference = 1st term.

24. The sum of 5 terms of an A.P. is 10, the sum of 17 terms is -17; find the series.

25. Find the A.P. in which the first 10 terms together = 100 and the next 10 terms = 300.

26. The 1st and last of $2n+1$ terms of an A.P. are a and c. Write down the sum and the middle term of the series.

27. The mth and nth terms of an A.P. are p, q. Find the 1st term and common difference.

28. The natural numbers 1, 2, 3, ... n^2 are arranged as a magic square, *i.e.* they are so placed in the compartments of a square that all the vertical columns, the horizontal rows and the diagonals have the same sum. Prove that this sum is $\frac{1}{2}n(n^2+1)$. Show that the first 16 numbers can be so arranged with each diagonal in A.P.

CHAPTER XLIII.

GEOMETRICAL PROGRESSION.

271. Numbers which proceed by a common factor are said to be in **Geometrical Progression.**

Thus 3, 12, 48, 192, ... form a G.P. in which the common ratio is 4,

1, -3, 9, -27, ... -3,

20, 10, 5, 2·5, ... $\frac{1}{2}$.

272. *To find the nth term of a* G.P

Let a be the 1st term, r the common ratio.

The 2nd term is ar,

3rd ..... ar^2,

4th ..... ar^3, and so on.

Thus the **nth term is ar^{n-1}.**

273. *To find the sum of n terms of a* G.P. *whose first term is a and common ratio r.*

Let s be the sum.

$$s = a + ar + ar^2 + \ldots + ar^{n-1},$$

$$sr = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n;$$

$\therefore$ by subtraction $s(1-r) = a(1-r^n)$;

$$\therefore\ s = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} = \frac{rl-a}{r-1}$$ where l is the nth term.

274. *To find the sum of an infinite number of terms of a* G.P.

[Infinity is represented by ∞.]

Let S_∞ denote the sum of an infinite number of terms

If r is not less than 1, the terms do not decrease as we proceed with the series;

$$\therefore\ S_\infty = \infty.$$

If r is a proper fraction, the terms decrease as we proceed, and it is possible to find a *limit* which the sum cannot exceed and to which it becomes indefinitely near as the number of terms is made infinitely great.

This *limit* is called the *sum to infinity.*

$$S_n = \text{the sum of } n \text{ terms} = \frac{a(1-r^n)}{1-r}.$$

$$S_\infty = \frac{a\{1-(\text{proper fraction})^\infty\}}{1-r} = \frac{a(1-0)}{1-r} = \frac{a}{1-r}.$$

e.g. $\dot{7} = \cdot 777 \ldots$ to infinity

$$= \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \ldots$$

$$= \frac{a}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}.$$

$\cdot 6\dot{2}\dot{3} = \cdot 62323 \ldots$ to infinity

$$= \frac{6}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \ldots = \frac{6}{10} + \frac{\frac{23}{1000}}{1-\frac{1}{100}}$$

$$= \frac{6}{10} + \frac{23}{990} = \frac{617}{990}.$$

The reason of the **rule for conversion of a mixed recurring decimal into a vulgar fraction** is made plain by treating an example in the way adopted for summing a G.P. (§ 273).

Let $s = \cdot 6\dot{2}\dot{3} = \cdot 62323 \ldots.$

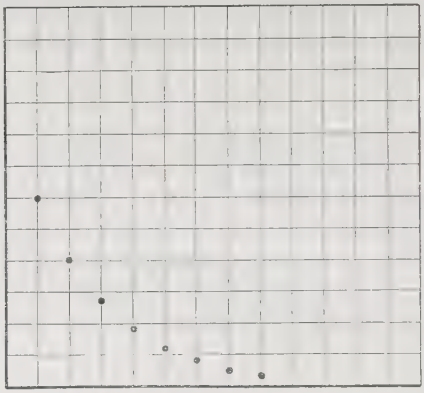
Then $10s = 6 \cdot 2323 \ldots$

and $1000s = 623 \cdot 2323 \ldots;$

$\therefore$ by subtraction $990s = 623 - 6;$

$$\therefore\ s = \frac{623-6}{990}.$$

275. Graphic representation of the successive terms of an infinite G.P.



Take n for abscissa, and ar^{n-1} for ordinate, giving to n the successive positive integral values.

If $a=6$, and $r=\frac{2}{3}$, the diagram shows approximately the successive terms as ordinates.

276. *To insert n geometric means between a and b.*

The 1st term is a, the last is b, and there are altogether $n+2$ terms;

$$\therefore\ b=ar^{n+1}.$$

From this we obtain the value of r; and the required means are ar, ar^2, $ar^3 \ldots ar^n$.

277. *To find a single geometric mean between a and b.*

Denote it by G.

Then a, G, b are in G.P.

$$\therefore\ \frac{\mathsf{G}}{a}=\text{the common ratio}=\frac{b}{\mathsf{G}};$$

$$\therefore\ \mathsf{G}^2=ab;$$

$\therefore\ \mathsf{G}=\sqrt{ab}$. This is called **the Geometric Mean** between a and b. It is obviously the same as the mean proportional, just as terms in G.P. are also in continued proportion.

Example 1. Sum the series $3\frac{3}{8} - 2\frac{1}{4} + 1\frac{1}{2} - \ldots$ to 7 terms.
It should be noticed that $(-1)^7 = -1$.

Here $$a = \frac{27}{8}, \quad r = \frac{-2\frac{1}{4}}{3\frac{3}{8}} = -\frac{2}{3}.$$

$$s = \frac{a(1-r^n)}{1-r} = \frac{27}{8} \cdot \frac{1-\left(-\frac{2}{3}\right)^7}{1-\left(-\frac{2}{3}\right)} = \frac{27}{8} \cdot \frac{1-\left(-\frac{2^7}{3^7}\right)}{1-\left(-\frac{2}{3}\right)}$$

$$= \frac{27}{8} \cdot \frac{1+\frac{2^7}{3^7}}{1+\frac{2}{3}} = \frac{27}{8} \cdot \frac{3}{5} \cdot \frac{3^7+2^7}{3^7}$$

$$= \frac{3^7+2^7}{40 \times 27} = \text{etc.}$$

Example 2. Sum the same series to 6 terms.

$$s = \frac{a(1-r^n)}{1-r} = \frac{27}{8} \cdot \frac{1-\left(-\frac{2}{3}\right)^6}{1-\left(-\frac{2}{3}\right)} = \frac{27}{8} \cdot \frac{1-\frac{2^6}{3^6}}{1+\frac{2}{3}}$$

$$= \frac{27}{8} \cdot \frac{3}{5} \cdot \frac{3^6-2^6}{3^6} = \frac{3^6-2^6}{40 \times 9} = \text{etc.}$$

Example 3. Sum the series $3\frac{3}{8} + 2\frac{1}{4} + 1\frac{1}{2} + \ldots$ to 6 terms.

Here $$r = \frac{2}{3}.$$

$$s = \frac{a(1-r^n)}{1-r} = \frac{27}{8} \cdot \frac{1-\left(\frac{2}{3}\right)^6}{1-\frac{2}{3}} = \frac{27}{8} \cdot \frac{\frac{3^6-2^6}{3^6}}{\frac{1}{3}}$$

$$= \frac{27}{8} \cdot \frac{3}{1} \cdot \frac{3^6-2^6}{3^6} = \frac{3^6-2^6}{72} = \text{etc.}$$

Examples. XLIII. a.

1. The 1st term of a G.P. is 3 and the common ratio 2, find the 5th term.
2. ,, 1st ..... is $\frac{1}{2}$ ..... 3, find the 6th ,,
3. ,, 2nd ..... is $\frac{1}{4}$ ..... $\frac{1}{2}$, find the 8th ,,
4. ,, 5th ..... is 243 ..... 3, find the 1st ,,
5. Sum the series $\frac{1}{4} + \frac{1}{2} + 1 + 2 + \ldots$ to 18 terms.
6. ..... $a + ab^2 + ab^4 + \ldots$ to x terms.
7. ..... $1 - c^3 + c^6 - \ldots$ to 10 terms.
8. ..... $\frac{1}{\sqrt{2}} - 1 + \sqrt{2} - \ldots$ to 10 terms.

9. Sum the series $\frac{1}{\sqrt{3}}+1+\sqrt{3}+\ldots$ to 8 terms.

10. ..... $5+3+1\frac{4}{5}+\ldots$ to 6 terms.

11. ..... $-3-6-12-\ldots$ to 9 terms.

12. In any G.P. the product of 1st and last term = the product of the 2nd and last but one.

13. Sum the series $9-6+4-\ldots$ to 7 terms.

14. Sum the same series to 6 terms.

15. Sum the series $\sqrt{3}-\sqrt{2}+\frac{2}{\sqrt{3}}-\ldots$ to 10 terms.

16. Sum n terms of the series whose rth term is $(-a)^r$.

17. The 2nd term is 6 and the 5th is 48, find the sum of 6 terms.

18. The 3rd term is 8 and the 6th is -1, find the sum of 7 terms.

19. Insert 3 geometric means between 5 and 80.

20. ..... 5 ..... 54 and $\frac{27}{32}$.

21. ..... 5 ..... 21 and $\frac{448}{243}$.

22. ..... 3 ..... 96 and 1536.

23. Find the geometric mean between 3 and 147.

24. ..... $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$.

25. ..... -2 and -1458.

26. Sum the series $1-1+1-1+\ldots$ to n terms.

27. Write down the 5th term of the series whose nth term is 2^n.

28. ..... 7th ..... 2^n-1.

29. ..... 1st ..... 3^n-2.

30. ..... 4th ..... $2+(-3)^n$.

31. ..... 1st .....

32. ..... 3rd .....

33. ..... $\overline{n+1}$|th .....3^n-1.

34. ..... $\overline{n-4}$|th ..... a^{n-2}.

35. ..... 1st ..... 2^{n-3}.

36. ..... 3rd ..... $\overline{n+1}$|th term is 2^n.

37. ..... $\overline{n-2}$|th .....

38. ..... $\overline{n+3}$|th ..... a^n-b.

Examples. XLIII. b.

1. Taking the number of the term for abscissa and the term for ordinate, plot the successive terms of 8, 6, 4, ... and of 8, 4, 2,....

State what happens with regard to the magnitude of the nth term in these series when n is made infinitely great.

Sum to infinity the following geometric series:

2. $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots$. 3. $9-6+4-\ldots$.

4. $4+3+\frac{9}{4}+\ldots$. 5. $16+2+\frac{1}{4}+$.

6. $\left(\frac{a}{b}\right)^{\frac{1}{2}}-\left(\frac{b}{a}\right)^{\frac{1}{2}}+\left(\frac{b}{a}\right)^{\frac{3}{2}}-\ldots$.

7. Sum to infinity $1+2x+3x^2+4x^3+\ldots$.

8. $1+3x+5x^2+7x^3+\ldots$.

9. Sum to n terms $1+3+7+15+\ldots+(2^r-1)+\ldots$.

Find the value in vulgar fractions of

10. $\dot{8}$. 11. $\cdot\dot{6}1\dot{2}$. 12. $\cdot1\dot{2}\dot{3}$. 13. $\cdot24\dot{1}\dot{2}$.

14. If P be the product of n terms in G.P., s their sum, s' the sum of their reciprocals, $P^2=\left(\frac{s}{s'}\right)^n$.

15. Prove that the insertion of geometric means between two numbers can be performed by inserting arithmetic means between their logarithms.

16. Insert 3 geometric means between 7 and 112.

17. ..... 2 ..... 2 and -54.

18. Sum the series $\cdot14+\cdot0028+\cdot000056+\ldots$ by writing down 8 or 9 terms and adding up. Sum it also as an infinite G.P. Compare results.

19. From 3 numbers in G.P. 3 others in G.P. are subtracted. If the remainders be in G.P., prove that all three series have the same common ratio.

20. The sum of the first two terms of a G.P. is 12, and of the first 3 terms 39; find the series, and determine if the conditions are satisfied by more than one series.

21. Which term of the series $5+5\sqrt{2}+10+\ldots$ is the same as the 200th term of $\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\ldots$?

22. Prove that the first 50 terms of $17+16\frac{2}{7}+16\frac{3}{7}+\ldots$ are together equal to the infinite series $200+120+72+\ldots$.

23. In the series $2+4+8+16+\ldots$ determine the number of digits in the 250th term by using logarithms.

24. If a, b, c be in G.P. and x, y be the arithmetic means between a, b and between b, c respectively, prove that $\frac{a}{x}+\frac{c}{y}=2$.

25. Sum the series $x^{b-2c}+x^{b-c}+x^b+\ldots$ to n terms.

26. Give the nth term of the series $1+5+13+29+\ldots$.

27. Find the sum of all the products formed by taking any two terms of an infinite G.P., and show that if this sum be $\frac{1}{2}$ the sum of the squares of the terms their common ratio is $\frac{1}{3}$.

28. If $1\frac{7}{9}$ and 1 are the 1st and 3rd terms of a G.P., find the sum to infinity.

CHAPTER XLIV.

HARMONICAL PROGRESSION.

278. Def. *If* $\frac{a}{c}=\frac{a-b}{b-c}$ *the quantities* a, b, c *are said to be in Harmonical Progression; and a series is said to be a Harmonical Progression when the above relation is satisfied by every three successive terms of it.*

Or, *three quantities are in* H.P. *when the first is to the third as the difference of the first and second is to the difference of the second and third.*

279. *If terms are in* H.P. *their reciprocals are in* A.P.

Let a, b, c be in H.P.

Then
$$\frac{a}{c}=\frac{a-b}{b-c};$$
$$\therefore\ ab-ca=ca-bc;$$
$$\therefore\ \frac{ab-ca}{abc}=\frac{ca-bc}{abc};$$
$$\therefore\ \frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a},$$

i.e. $\frac{1}{a},\ \frac{1}{b},\ \frac{1}{c}$ are in A.P.

280. From the property proved in the last section we can write down any term of an H.P. whose first three terms are given, by inverting the terms, applying the methods of A P., and inverting the results.

The insertion of means can be performed in this way.

It must be carefully observed that this method does not give us any formula for summing a series in H.P. Summation may be performed, if necessary, by writing down the terms as above and **adding them up.**

281. *To find the Harmonic Mean between* x *and* y.

Denote it by H.

Then x, H, y are in H.P;

$$\therefore \frac{x}{y} = \frac{x - \mathsf{H}}{\mathsf{H} - y} \text{ (by definition)};$$

$$\therefore x\mathsf{H} - xy = xy - y\mathsf{H};$$

$$\therefore \mathsf{H} = \frac{2xy}{x+y}.$$

A, the Arithmetic Mean, has been found to be $\frac{x+y}{2}$.

G, the Geometric ....., ... $\sqrt{xy}$.

H, the Harmonic ....., ... $\frac{2xy}{x+y}$.

A is familiarly known as the *average* of x and y. G, H might be called the Geometric and Harmonic average.

$$\mathsf{AH} = \frac{x+y}{2} \cdot \frac{2xy}{x+y} = xy = \mathsf{G}^2;$$

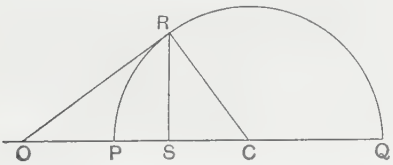
$$\therefore \frac{\mathsf{G}}{\mathsf{A}} = \frac{\mathsf{H}}{\mathsf{G}},$$

i.e. A, G, H form a G.P. or continued proportion.

Also $\mathsf{A} - \mathsf{G} = \frac{x+y-2\sqrt{xy}}{2} = \frac{(\sqrt{x}-\sqrt{y})^2}{2}$ = a positive quantity.

$\therefore$ A, G, H form a *descending* series.

282. *To represent* A, G, H *graphically.*



In a str. line OPQ mark off OP, OQ to represent quantities whose mean is required. Bisect PQ at C. On PQ describe a semicircle PRQ. Draw OR a tangent, and RS perp. to PQ. OC, OR, OS represent respectively the Arithmetic, Geometric and Harmonic Means.

Examples. XLIV.

Continue to 6 terms the series

1. $\frac{1}{2}+\frac{4}{9}+\frac{2}{5}+\ldots$. **2.** $\frac{1}{6}+\frac{2}{11}+\frac{1}{5}+\ldots$. **3.** $1+\frac{3}{8}+\frac{3}{13}+\ldots$.

Find the 12th term of

4. $\frac{3}{2}, \frac{12}{7}, 2\ldots$. **5.** $-\frac{1}{5}, -1, +\frac{1}{3}\ldots$. **6.** $\frac{3}{8}, \frac{1}{2}, \frac{3}{4}\ldots$.

7. Insert 4 harmonic means between 1 and 6.

8. ..... 3 ..... 1 and 20.

9. ..... 4 ..... 2 and 12.

10. ..... 3 ..... 9 and $4\frac{1}{5}$.

11. Find the harmonic mean between 2 and $\frac{6}{5}$.

12. If $a-b$, $a-c$, $a-d$ are in H.P., prove that the following three series of quantities are equally so:

(i) $b-c, b-d, b-a$.

(ii) $c-d, c-a, c-b$.

(iii) $d-a, d-b, d-c$.

13. To each of three consecutive terms in G.P. the middle one is added. Prove that the three results are in H.P.

14. If a, b, c are in H.P., then $\frac{1}{a-b}+\frac{1}{b-c}+\frac{4}{c-a}=\frac{1}{c}-\frac{1}{a}$.

CHAPTER XLV.

MISCELLANEOUS SERIES AND MATHEMATICAL INDUCTION.

283. *Sum of the first n natural numbers* (s_1).

These form an A.P.

$$\therefore\ s_1=1+2+3+\ldots+n=\frac{n}{2}(n+1)\quad\left[s=\frac{n}{2}(a+l)\right].$$

284. *Sum of the squares of the first n natural numbers* (s_2).

$$n^3-(n-1)^3=3n^2-3n+1,$$
$$(n-1)^3-(n-2)^3=3(n-1)^2-3(n-1)+1,$$
$$(n-2)^3-(n-3)^3=3(n-2)^2-3(n-2)+1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$2^3-1^3=3\,.\,2^2-3\,.\,2+1,$$
$$1^3-0^3=3\,.\,1^2-3\,.\,1+1;$$
$$\therefore\ \text{by addition } n^3=3s_2-3s_1+n;$$

$$\therefore\ 3s_2 = n^3 - n + 3s_1 = n(n^2-1) + \frac{3n(n+1)}{2}$$

$$= \frac{n\,.\,(n+1)}{2}\{2(n-1)+3\} = \frac{n(n+1)(2n+1)}{2};$$

$$\therefore\ s_2 = \frac{n(n+1)(2n+1)}{6}.$$

285. *Sum of the cubes of the first n natural numbers* (s_3).

$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1,$$

$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$

.....

$$2^4 - 1^4 = 4\,.\,2^3 - 6\,.\,2^2 + 4\,.\,2 - 1,$$

$$1^4 - 0^4 = 4\,.\,1^3 - 6\,.\,1^2 + 4\,.\,1 - 1;$$

$$\therefore \text{ by addition } n^4 = 4s_3 - 6s_2 + 4s_1 - n;$$

$$\therefore\ 4s_3 = n^4 + n + 6s_2 - 4s_1$$

$$= n^4 + n + n(n+1)(2n+1) - 2n(n+1)$$

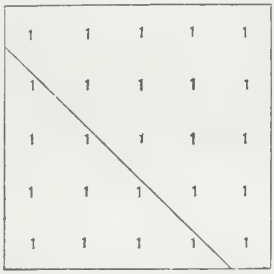
$$= n(n+1)\{n^2 - n + 1 + 2n + 1 - 2\}$$

$$= n^2(n+1)^2;$$

$$\therefore\ s_3 = \left[\frac{n(n+1)}{2}\right]^2.$$

286. The foregoing series (as well as the series of odd numbers) may be simply summed by a method which appeals to the sight.

To sum $1 + 2 + 3 + \ldots + n$.



In the diagram the columns above the line parallel to the diagonal form the series to n terms (supposing that there are n columns). Those below the line form the series to $n - 1$ terms.

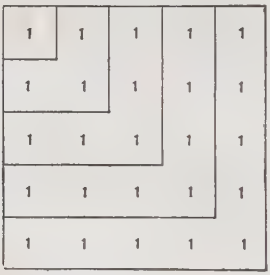
Call these s, s'.

Then $s + s' = n^2$,

$s - s' =$ the diagonal line of figures $= n$;

$\therefore$ by addition $2s = n(n+1)$; $\therefore$ $s = \dfrac{n(n+1)}{2}$.

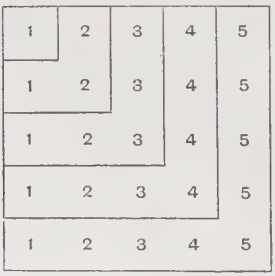
To sum $1 + 3 + 5 + \ldots$ to n terms.



The partitions in the diagram are seen to include the odd numbers.

$\therefore$ the sum of the first n odd numbers $= n^2$.

To sum $1^2 + 2^2 + 3^2 +$ etc. $\ldots + n^2$ (s_2).



The r^{th} compartment contains $1 + 2 + \ldots + r-1 + r^2$,

i.e. $\dfrac{r(r-1)}{2} + r^2$,

i.e. $\dfrac{3r^2}{2} - \dfrac{r}{2}$;

$\therefore$ $\dfrac{3s_2}{2} - \dfrac{1+2+3+\ldots+n}{2}$

$=$ sum of all the numbers in the compartments

$= (1 + 2 + 3 + \ldots + n)n$;

$$\therefore \quad \frac{3s_2}{2} = (1+2+\ldots+n)\,(n+\tfrac{1}{2}) = \frac{n\,.\,\overline{n+1}\,.\,\overline{2n+1}}{4};$$

$$\therefore \quad s_2 = \frac{n(n+1)(2n+1)}{6}.$$

To sum $1^3+2^3+3^3+\ldots+n^3$.

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

The r^{th} compartment contains r^3.

[For it contains $2r(1+2+3+\ldots+r)-r^2$.]

$$\therefore \quad 1^3+2^3+3^3+\ldots+n^3$$
$$= \text{the sum of all the numbers}$$
$$= s_1+2s_1+3s_1+\ldots$$
$$= s_1{}^2 \quad [\text{where } s_1 = 1+2+3+\ldots+n].$$

For further developments of this method see *Nouvelles Annales* for 1870.

287. *Sum the series* $a+(a+b)x+(a+2b)x^2+(a+3b)x^3+\ldots$ *to n terms.*

Let s denote the sum.

$$s = a+(a+b)x+(a+2b)x^2+(a+3b)x^3+\ldots+(a+\overline{n-1}\,.\,b)x^{n-1};$$

$$\therefore \quad sx = \qquad ax+(a+\ \ b)x^2+(a+2b)x^3+\ldots+(a+\overline{n-2}\,.\,b)x^{n-1}$$
$$+(a+\overline{n-1}\,.\,b)x^n.$$

$\therefore$ by subtraction

$$s(1-x) = a+bx+bx^2+bx^3+\ldots+bx^{n-1}-(a+\overline{n-1}\,.\,b)x^n$$
$$= a+\frac{bx(1-x^{n-1})}{1-x}-ax^n-(n-1)bx^n$$
$$= a(1-x^n)-(n-1)bx^n+\frac{bx(1-x^{n-1})}{1-x};$$

$$\therefore\ s=\frac{a(1-x^n)-(n-1)bx^n}{1-x}+\frac{bx(1-x^{n-1})}{(1-x)^2}.$$

If x is a proper fraction the sum to infinity is

$$\frac{a}{1-x}+\frac{bx}{(1-x)^2}.$$

Example 1. Find $\sum_{r=1}^{r=n}(3r^2+5r)$,

i.e. sum to n terms the series whose rth term is $3r^2+5r$.

$$\begin{aligned}
&\text{1st term}=3\,.\,1^2+5\,.\,1,\\
&\text{2nd term}=3\,.\,2^2+5\,.\,2,\\
&\quad\text{etc. ....}\\
&\therefore\ \text{the sum}=3(1^2+2^2+3^2+\ldots+n^2)+5(1+2+3+\ldots+n)\\
&\qquad=3\,.\,\frac{n(n+1)(2n+1)}{6}+5\,.\,\frac{n(n+1)}{2}\\
&\qquad=\frac{n(n+1)}{2}(2n+1+5)=n(n+1)(n+3).
\end{aligned}$$

Example 2. Sum to n terms $1\,.\,4\,.\,5+2\,.\,5\,.\,6+3\,.\,6\,.\,7+\ldots$.

The nth term $=n(n+3)(n+4)=n^3+7n^2+12n$.

$\therefore$ the sum

$$\begin{aligned}
&=1^3+2^3+3^3+\ldots+n^3+7(1^2+2^2+3^2+\ldots+n^2)+12(1+2+3+\ldots+n)\\
&=\frac{n^2(n+1)^2}{4}+\frac{7n(n+1)(2n+1)}{6}+\frac{12n(n+1)}{2}\\
&=\frac{n(n+1)}{2}\left\{\frac{n^2+n}{2}+\frac{14n+7}{3}+12\right\}\\
&=\frac{n(n+1)}{2}\,.\,\frac{3n^2+3n+28n+14+72}{6}\\
&=\frac{n(n+1)(3n^2+31n+86)}{12}.
\end{aligned}$$

PILES OF SHOT.

288. *To find the number of shot in a complete pyramidal pile of n layers on a triangular base.*

The top layer consists of 1 ball,
the 2nd ..... $1+2$ balls,
the 3rd ..... $1+2+3$....

The r^{th} layer from the top consists of $1+2+3+\ldots r$, *i.e.* $\frac{r(r+1)}{2}$.

$\therefore$ we have to find $\Sigma\frac{1}{2}(r^2+r)$ from $r=1$ to $r=n$.

This expression $= \frac{1}{2}(1^2 + 2^2 + 3^2 + \ldots + n^2) + \frac{1}{2}(1 + 2 + 3 + \ldots + n)$

$$= \frac{1}{2} \cdot \frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6} + \frac{1}{2} \cdot \frac{n \cdot \overline{n+1}}{2}$$

$$= \frac{n \cdot \overline{n+1}}{12} \{2n + 1 + 3\} = \frac{n \cdot \overline{n+1} \cdot \overline{n+2}}{6}.$$

289. *To find the number of shot in a complete pyramidal pile n layers of shot on a square base.*

The top layer consists of 1,
the 2nd ..... 2^2,
the 3rd ..... 3^2,
the rth ..... r^2.

$\therefore$ the total number $= 1^2 + 2^2 + 3^2 + \ldots + n^2$

$$= \frac{n \cdot \overline{n+1} \cdot 2n+1}{6}.$$

290. *To find the number of shot in a completed pile arranged on a rectangular base.*

Let the number of shot in the base be m along the length and n along the breadth.

The number in the base is mn.

..... in the layer next above is $(m-1)(n-1)$,

and at the top the pile finishes with a row of $m - n + 1$ shot.

The number of layers is n.

The total number

$= 1(m-n+1) + 2(m-n+2) + 3(m-n+3) + \ldots$ to n terms

$= (m-n)(1 + 2 + 3 + \ldots + n) + 1^2 + 2^2 + 3^2 + \ldots + n^2$

$$= (m-n)\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n}{6}(n+1)\{3m - 3n + 2n + 1\} = \frac{n}{6}(n+1)(3m - n + 1).$$

Examples. XLV. a.

1. Find the number of shot in a completed triangular pile of 8 courses.
2. .... 10 .....
3. .... 15 .....
4. Find the number of shot in a completed square pile of 9 courses.
5. .... 12 .....

6. If the base is a rectangle of length 10 and breadth 7 shot, what is the number of shot in the top row if the pile is complete?

7. What is the number in the pile?

8. If a triangular pile has 6 courses, the base having 9 shot in a side, what is the number in the incomplete pile?

9. If there are 64 shot in the base of a square pile, what is the number in the 4 lowest courses?

10. In a triangular pile how many shot are there in the 7th layer counting from the top? How many shot are there above this layer?

11. The top of a rectangular pile is a line of 6 shot, and the bottom layer contains 66; how many layers are there; and how many shot in all?

12. The lowest course of a square pile contains 49 shot, and there are 3 layers; how many shot are required to complete it?

13. Find the number in an incomplete square pile, the top layer containing 529, and the bottom 5184.

Examples. XLV. b.

1. Three quantities a, b, c are in A.P., G.P. or H.P. according as
$$\frac{a-b}{b-c}=\frac{a}{a} \text{ or } \frac{a}{b} \text{ or } \frac{a}{c}.$$

2. If A, G, H be the Arithmetic, Geometric, and Harmonic mean between a and b, express A, G, H geometrically. Prove both geometrically and algebraically that $G^2=AH$ and that $A>G>H$.

3. If $H=\frac{2ab}{a+b}$, prove a, H, b are in H.P.

4. Insert 3 harmonic means between $\frac{1}{2}$ and $\frac{1}{10}$.

5. Sum to n terms the series whose n^{th} term is n^2+n.

6. If $b+c$, $c+a$, $a+b$ are in H.P., a^2, b^2, c^2 are in A.P.

7. If a, b, c, d are in G.P., $(a+b+c)(a-b)=a(a-d)$.

8. The sum of 4 numbers in A.P. is -24, and the product is 945: find them.

9. If a, b, c are in H.P. $\frac{2}{b}=\frac{1}{b-a}+\frac{1}{b-c}$.

10. a, b, c are in H.P., and half the middle term is subtracted from each, what is the resulting progression?

11. How many terms of 19, 15, 11, ... amount to 54?

12. a is the first term, d the common difference, $2n+1$ the number of terms. What is the middle term?

13. Find the sum of $a^2+(a+d)^2+(a+2d)^2+\ldots$ to n terms.

14. Find the sum of all integers of 3 figures which are divisible by 13.

15. Find the sum of all the numbers in a multiplication table which goes as far as 12×12.

16. Sum the series $1^2-2^2+3^2-4^2+\ldots$ to 21 terms.

17. Sum to n terms the series whose n^{th} term is n^2-1.

18. A man travels 21 miles the 1st day, and 3 miles less each succeeding day. When will he reach a point 75 miles distant from the start? Explain the double answer.

19. Which term of the series 7, 11, 15, ... is 107?

20. Sum the squares of the odd terms of the series

$$a+ar+ar^2+\ldots+ar^{2n-1}.$$

21. Sum to n terms the series whose n^{th} term is n^3-n.

22. ..... $3n^2-n$.

23. ..... n^2+3n+2.

24. ..... an^2+bn.

25. ..... $\frac{n^2}{2}-\frac{1}{6}$.

26. Find the series whose sum to n terms is an^2.

27. Show that the A.M., G.M. and H.M. between x and y are in G.P. If the common ratio of this G.P. be $\frac{3}{5}$ find the ratio $x:y$.

28. Write down the n^{th} terms of the following series

$$1^2+2^2+3^2+\ldots.$$

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots,$$

and shew that they form neither arithmetic nor geometric progressions.

29. Sum $\frac{5}{6}+\frac{5}{9}+\frac{1}{2}\frac{0}{7}+\ldots$ to n terms and to infinity.

30. $6-5+\frac{2}{6}\frac{5}{5}-\ldots$ to $2n$ terms and to infinity.

31. $1-3+5-7+\ldots$ to $2n$ terms.

32. $x^2+(x+1)^2+(x+2)^2+\ldots$ to n terms.

33. $12^2+13^2+14^2+\ldots$ to 10 terms.

34. Prove $bc-ad$ is positive, zero or negative according as a, b, c, d are in A.P., G.P. or H.P.

35. If S_n is the sum of n terms of $a+d, a+2d, \ldots$, find the value of $S_{n+4}-2S_{n+3}+S_{n+2}$.

36. If b is half the harmonic mean between a and c, then

$$a^3-b^3+c^3+3abc=(a-b+c)^3.$$

37. Sum the series $1-2x+3x^2+4x^3+\ldots$ to n terms. Test the result.

38. If the arithmetic mean between two numbers $=1$, the harmonic mean is the square of the geometric mean.

39. The sum of the squares of the first n odd integers $=\frac{n}{3}(2n-1)(2n+1)$.

40. If between two given numbers there be inserted 2 Arithmetical means A_1, A_2, 2 Geometric means G_1, G_2, and 2 Harmonic means H_1, H_2, then

$$\frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}.$$

41. If a, b, c are in A.P., prove that

$$\frac{2}{9}(a+b+c)^3=a^2(b+c)+b^2(c+a)+c^2(a+b).$$

42. Sum to n terms the series whose n^{th} term is $2^{2n}-2n$.

43. The difference of two numbers is 48, and their arithmetic mean exceeds their geometric by 18. Find them.

44. Find three numbers in G.P. such that their sum is 14 and their product 64.

45. In an A.P. the sum of $n+1$ terms $=(n+1)(n+\frac{4}{5})$; find the n^{th} term.

46. In the series 1, 3, 5, ... to $2n$ terms prove that the ratio of the 2nd half to the 1st half is a constant for all values of n.

47. If the harmonic mean between x and y be n times the geometric, then $x : y = 1 \pm \sqrt{1-n^2} : 1 \mp \sqrt{1-n^2}$.

48. If a be the arithmetic mean between b and c, and c the harmonic mean between a and b, then b is the geometric mean between a and c.

49. Find the n^{th} term and the sum of n terms of the series

$$1+3+6+10+15+21+\dots.$$

[Notice that the 2nd term is $1+2$, 3rd term $1+2+3$, and so on.]

50. Sum to n terms the series

(i) $1.2+2.3+3.4+\dots.$

(ii) $1.3+3.5+5.7+\dots.$

(iii) $1.2.3+2.3.4+3.4.5+\dots.$

51. If a, b, c, d are in G.P., prove that

$$(a-b+c-d)^2=(a-b)^2+2(b-c)^2+(c-d)^2.$$

52. Sum to n terms the series

(i) $a+(a+1)x+(a+2)x^2+\dots.$

(ii) $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\dots.$

(iii) $1.3.5+3.5.7+5.7.9+\dots.$

53. Sum $\frac{1}{2}-\frac{2}{2^2}+\frac{3}{2^3}-\frac{4}{2^4}+\dots$ to infinity.

54. Write down the n^{th} term of the following series and so sum them to n terms:

(i) $1^2.3+2^2.5+3^2.7+\dots.$

(ii) $1.4+4.7+7.10+\dots.$

55. If a, b, c are in H.P., $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ are in A.P., prove this and the converse.

MATHEMATICAL INDUCTION.

291. An example will explain the method known by this name. To prove that the sum of the first n natural numbers

$$=\frac{n(n+1)}{2}.$$

Suppose it true for n terms,

$$i.e.\ \text{suppose } 1+2+3+\ldots+n=\frac{n(n+1)}{2}.$$

By adding $n+1$ to both sides we get

$$\text{the sum of } n+1 \text{ terms}=\frac{n(n+1)}{2}+n+1=\frac{(n+1)(n+2)}{2}.$$

$\therefore$ if the formula holds for any number of terms, it holds for the next higher number of terms.

But it is evidently true for 2 terms, $\therefore$ for 3, and consequently for 4 terms, and so on.

Hence it is true for *any* number of terms.

Examples. XLV. c.

Prove the following by Mathematical Induction :

1. The sum of n terms of an A.P. $=\frac{n}{2}\{2a+(n-1)d\}$.

2. $1+3+5+\ldots$ to n terms $=n^2$.

3. a^n-b^n is divisible by $a-b$.

4. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots$ to n terms $=1-\frac{1}{n+1}$.

5. $1^2+2^2+3^2+\ldots$ to n terms $=\frac{n(n+1)(2n+1)}{6}$.

6. $1^3+2^3+3^3+\ldots$ to n terms $=\frac{n^2(n+1)^2}{4}$.

7. $1.2+2.3+3.4+\ldots$ to n terms $=\frac{n(n+1)(n+2)}{3}$.

8. The sum of n terms of a G.P. $=\frac{a(1-r^n)}{1-r}$.

9. If n be even, n^2+2n is divisible by 8.

10. If $\phi(n)=n^3+20n$, find $\phi(n+2)-\phi(n)$, and hence prove that n^3+20n is divisible by 48 if n be even.

11. Prove that $1.3+2.4+3.5+\ldots$ to n terms $=\frac{1}{6}n(n+1)(2n+7)$.

12. The amount of £P at compound interest in n years at x per cent. per annum $=\mathrm{PR}^n$, where $\mathrm{R}=1+\frac{x}{100}$.

13. The difference between any integer and its cube is a multiple of 6.

14. $3^{4n}-1$ is divisible by 80.

CHAPTER XLVI.

REVISION PAPERS.

XLVI. a.

1. Simplify $\dfrac{4\sqrt{2}-2\sqrt{3}}{2\sqrt{18}-\sqrt{27}}$.

2. Solve $\sqrt{x+a}+\sqrt{x-a}=\sqrt{2a}$.

3. The sum of three numbers in A.P. is 33 and their product 792. Find them.

4. In a G.P. the sum of the first 3 terms : the sum to infinity $= 7 : 8$. Find the common ratio.

5. If $\dfrac{x}{a+2b+c}=\dfrac{y}{2a+b-c}=\dfrac{z}{4a-4b+c}$, then

$$\frac{a}{x+2y+z}=\frac{b}{2x+y-z}=\frac{c}{4x-4y+z}.$$

6. Find the sum of n terms of a series whose r^{th} term is $ar+b$.

7. If the arithmetic mean between x and y = twice their geometric mean, then $\dfrac{x}{y}=\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$ or the reciprocal of this.

XLVI. b.

1. Find $(\sqrt{3+2\sqrt{2}})^{-1}$ to 4 places.

2. Solve $\sqrt{x^2-6x+17}+2x^2=12x-13$.

3. If $\dfrac{a+b}{ax+by}=\dfrac{b+c}{bx+cy}=\dfrac{c+a}{cx+ay}$, each ratio $=\dfrac{2}{x+y}$, if $a+b+c$ is not zero.

4. The sum of 3 numbers in H.P. is 11, and the sum of their squares is 49. Find the numbers.

5. In an A.P. the 1st term is 7, the common difference 2, and the sum of n terms 247 ; find n.

6. Find the product $a \,.\, a^2 \,.\, a^3 \ldots a^n$.

7. Find by logarithms $4^{\frac{1}{3}} \div 2^{\frac{1}{5}}$, and simplify

$$\log \cdot 06 + \log(\cdot 6)^2 - \log 4 - \log 54.$$

XLVI. c.

1. If z is a mean proportional between x and y,

$$\frac{x}{y}=\frac{x^2+xz+z^2}{y^2+yz+z^2}.$$

2. If the illumination from a light varies inversely as the square of the distance from the light, show that for all points in a straight line between 2 equal lights the total illumination is least midway between them.

3. $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$; prove that

$$(b-c)x+(c-a)y+(a-b)z=0.$$

4. Find a series in which the sum of n terms shall be $3n^2$ for all values of n.

5. Find the product $a^{\frac{1}{2}}a^{\frac{1}{4}}a^{\frac{1}{8}}\ldots$ to infinity.

6. Sum to n terms the series whose r^{th} is 2^r-r.

7. How many digits are there in the 101st term of $1+5+25+125+\ldots$?

XLVI. d.

1. Find 4 numbers in A.P. such that the product of the extremes shall be 45, and the product of the means 77.

2. Find the difference of the roots of the equation

$$x^2-2bx+c^2=0.$$

Write down an equation whose roots are each equal to the sum of the reciprocals of the roots of this equation.

3. A cask was filled with wine and water mixed in the ratio 5 : 3. When 8 gallons had been drawn off and the cask filled up with water the ratio was 3 : 5. How many gallons did the cask hold?

4. The first and last of 46 terms in A.P. are -5 and $+25$: find the two middle terms and the sum of all the terms.

5. Find by logarithms the smallest number of terms of the G.P. 4, 6, 9 ... which have a sum exceeding 8000.

6. If an elastic ball, let fall on a table, always rebounds to a height which is the height from which it falls multiplied by e^2 (e being less than 1), find the total space described by the ball if it is let fall originally from a height h and left to itself.

7. If a, b, c are in H.P., $a(b-c)x^2+b(c-a)xy+c(a-b)y^2$ is a perfect square.

XLVI. e.

1. If $\alpha+\beta=-p$, and $\alpha\beta=q$ find in terms of p and q the equation whose roots are $\alpha+2\beta$ and $\beta+2\alpha$.

2. At present B's age : A's age $=5:2$, but in 30 years the ratio will be 35 : 23. Find their ages.

3. Solve to 2 places of decimals

$$5^x.\,7^{x-1}=11^{x+2}.$$

4. Sum to n terms $2^2+4^2+6^2+\ldots$.

5. Find an infinite G.P. in which each term is 3 times the sum of all that follow it.

6. Three regular polygons of p, q, r sides respectively fit round a point exactly: prove that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{1}{2}.$$

7. Prove that $\frac{a+2c}{b+2d}$ is intermediate in value between $\frac{a}{b}$ and $\frac{c}{d}$. Illustrate this also by a figure.

XLVI. f.

1. Nine numbers $\begin{matrix} a & b & c \\ d & e & f \\ g & h & k \end{matrix}$ are arranged in a square so that each horizontal line, each vertical line, and each diagonal line has a sum equal to x. Prove that $e=\frac{x}{3}$, and $a+c+k+g=\frac{4x}{3}$.

2. On a cylinder, whose length is 6 inches and circumference 3 inches, is a screw whose thread makes 4 turns in the length; on another cylinder exactly equal a thread makes 3 turns. Find the ratio of the lengths of the threads.

3. If $3c=\sqrt[3]{100}+2\sqrt[3]{10}+3$, prove $3(c-1)^3=20c$.

4. Test which is the greater $\sqrt{17}$ or $2\sqrt[4]{18}$.

5. Find by logarithms $(66901\times 337\div 7824)^{\frac{1}{5}}$.

6. Prove $\log 20+7\log\frac{1}{1}\frac{5}{6}+5\log\frac{2}{2}\frac{4}{5}+3\log\frac{8}{8}\frac{0}{1}=1$.

7. The circumference of a circle varies as the diameter, and when the diameter is $\frac{1}{2}$ foot the circumference is 1·5708 feet. Find to 3 places the length of an arc subtending 3° at the centre of a circle of 50 feet diameter.

XLVI. g.

1. The sum of all the products in a multiplication table extending as far as $n\times n=\left(\frac{n\,.\,n+1}{2}\right)^2$.

2. The angles of a triangle are in A.P. What is the middle one?
..... pentagon .....

3. The natural numbers are grouped thus; 1; 2, 3; 4, 5, 6; etc. ....
Prove that the sum of the n^{th} group $=\frac{n(n^2+1)}{2}$.

4. Find by logarithms an approximate value for the irrational root of the equation $3^{2x}-5\times 3^x+6=0$.

5. If x is real $\frac{x^2-4x-20}{x-7}$ cannot have a real value between 8 and 12.

6. Along a straight line mark off lengths in succession 2 inches, 1 inch, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc. ..., and by measurement sum the series to 7 terms. Verify the result by algebra.

7. Draw the graph of $\sqrt{x}+\frac{1}{2}\sqrt{y}=1$, and find from the figure where it is met by the line $2x+\frac{1}{2}y=1$. How are these lines related?

XLVI. h.

1. Between what powers of 10 do the following lie: 37^2, 37^3, 37^5 $37^{\frac{1}{10}}$, $37^{\frac{1}{100}}$.

2. Arrange in order of magnitude $5^{\frac{1}{6}}$, $6^{\frac{1}{6}}$, $7^{\frac{1}{7}}$; and find their product to 2 places of decimals.

3. Find the least positive value of $4x+\frac{9}{x}$.

4. The sum of 5 terms of an A.P. is 10, and the sum of 17 terms is -17; find the series.

5. The plan of an estate is drawn to a scale of 10 inches to a mile. Find the area of the part which represents 20 acres.

6. How many integers between 200 and 300 are divisible by 7? Find their sum.

7. If the time of travelling any distance $\propto \sqrt{\text{distance}}$, and the time for 144 feet be 3 seconds, find the time for 400 feet.

XLVI. i.

1. Find from the tables the logarithms of 400, 401, 402, 403, 404, 405, 406. Find the excess of each over log 400. On squared paper represent the increments of the logarithms corresponding to the addition of 1, 2, 3, 4, 5, 6 to the number 400. Show how to find, with the help of your diagram, log 403·2, log 403·5, log 403·3.

2. Express $(\sqrt{\frac{3}{2}} - \frac{2}{3}\sqrt{\frac{2}{3}}) \div (\sqrt{6} + \sqrt{\frac{2}{3}})$ as a ratio of 2 whole numbers.

3. Two companies appeared in the field in strength as 9 to 11. At the end of the day the relative strength was 5 to 8. Of the men in the companies 35 per cent. were missing, and 30 men of these belonged to the 2nd company. Find the number in each company at first.

4. Insert 5 arithmetic means between $a-2b$ and $3a+b$.

5. The sum of 7 terms of an A.P. is 147, and the product of the 1st and last is 297. Find the common difference.

6. The geometric mean of two numbers is $\frac{1}{2}$, and the harmonic mean $\frac{2}{5}$. Find the numbers.

7. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^2-b^2+c^2-d^2}{(a-b)^2+(c-d)^2}=\frac{(a+c)^2-(b+d)^2}{(a-b+c-d)^2}$.

XLVI. k.

1. What number must be subtracted from each of the numbers 8, 10, 13, 17, that the remainders may be in proportion?

2. Find the number of digits in the product of 2^{19} by 3^{17}, and find the first 2 figures of the product.

3. In a series of right-angled triangles, one side has, in succession, the lengths 1×4 in., 2×6 in., 3×8 in., 4×10 in., etc., and the hypotenuse differs in length from this side by 1 inch: show that the lengths of the other side form an A.P.

4. Find the square root of $3x-1+2\sqrt{2x^2+x-6}$.

5. If the m^{th} term of a harmonic series be n, and the n^{th} term m, prove that the $(m+n)^{\text{th}}$ term is $\frac{mn}{m+n}$.

6. Show that the arithmetic means between a and b, between $\sqrt{\frac{a}{b}}$ and $\sqrt{\frac{b}{a}}$, and between $\frac{1}{a}$ and $\frac{1}{b}$ are in G.P.

7. The sum of 3 numbers in H.P. is 11, and the sum of their squares is 49. Find them.

XLVI. l.

1. The sides of a right-angled triangle are in A.P. Show that they are proportional to 3, 4, 5.

2. Prove that any integral cube is the difference of two integral squares.

3. Solve $15^{2x} \cdot 5^{x-4} \cdot 11^{x-2} = 7^{x-1}$.

4. If n is odd, the sum of $1+2^2+3+4^2+5+6^2+\ldots$ to n terms
$$= \tfrac{1}{12}(n+1)(2n^2+n+3).$$

5. If P, Q, R are the p^{th}, q^{th}, r^{th} terms of an A.P. prove that
$$\mathrm{P}(q-r)+\mathrm{Q}(r-p)+\mathrm{R}(p-q)=0.$$

6. If in the last question the series were a G.P. the result would be $\mathrm{P}^{q-r} \cdot \mathrm{Q}^{r-p} \cdot \mathrm{R}^{p-q}=1$. Deduce this result from the former by logarithms.

7. The area of a $\triangle$ is $\sqrt{s(s-a)(s-b)(s-c)}$ where s = semiperimeter. Find by logarithms the area of a $\triangle$ whose sides are 606, 121, 725 yards.

XLVI. m.

1. The incomes of A and B are as 5 to 3, and their expenditures as 7 to 4. Each saves £300 a year. Find their incomes.

2. Which term of the series 1, 3, 5, 7 ... is a mean proportional between the 23rd and 63rd terms?

How many terms of the same series have their sum equal to one fourth of the sum of the first 20 terms?

3. The perimeter of a right-angled $\triangle$ is 15 feet, and the hypotenuse exceeds one of the sides by $2\frac{1}{2}$ feet. Find the 3 sides.

4. Find the sum of 9 terms of the series
$$2a+a\sqrt{2}+a+\frac{a}{\sqrt{2}}+\ldots.$$
What value must a have so that the sum *ad inf.* may be 8?

5. Prove $\log_a b \times \log_b a = 1$. Find $\log_4 1000$ from $\log_{10} 2$.

6. 105 halfpenny pieces lying on a flat surface with their edges in contact are just contained by a frame in the form of an equilateral $\triangle$. The diameter of a halfpenny being 1 inch, show that the side of the $\triangle$ is $(13+\sqrt{3})$ inches.

7. A man starts on a journey of 9 miles at 3 miles an hour, and increases his pace by $\frac{1}{4}$ mile an hour each quarter of an hour. How long does he take?

CHAPTER XLVII.

SCALES OF NOTATION.

292. In the ordinary notation of Arithmetic the value of a digit depends on its position with regard to the other digits of the same number.

Thus in the number 5837 there are 7 units, 3 tens, 8 hundreds and 5 thousands. In the number ·614 there are 6 tenths, 1 hundredth and 4 thousandths.

The number 5837·614 might be written in the form

$$5\times10^3+8\times10^2+3\times10+7+\frac{6}{10}+\frac{1}{10^2}+\frac{4}{10^3}.$$

This is the common scale of notation, the **decimal or denary scale**, and 10 is called the **radix** of the scale. Any other number might be employed instead of 10 for the radix of a scale.

If the scale of notation were r, then 5837·614 would mean

$$5r^3+8r^2+3r+7+\frac{6}{r}+\frac{1}{r^2}+\frac{4}{r^3}.$$

In the scale of 10 the number ·614 is called a decimal fraction. In any other scale it must be called a **radix fraction**.

In any scale the largest digit which may occur is less by 1 than the radix of the scale. In the scale of 12, known as the duodenary scale, ten and eleven may occur as well as the ordinary digits; and it is necessary to denote them by single symbols. Thus t denotes 10, e 11.

If 263 denotes a number in the scale of 8, the number is equivalent to $2\times8^2+6\times8+3$, *i.e.* 179 in the denary scale.

293. *To convert a number into the scale of r.*

Let N be the number.

Let e, d, c, b, a be the digits, in the order in which they occur from left to right, in the new scale.

Then $\mathsf{N}=er^4+dr^3+cr^2+br+a$.

Divide through by r.

Let N_1 be the integral quotient when N is divided by r.

Then $\mathsf{N}_1=er^3+dr^2+cr+b$, remainder a.

Let N_2 be the integral quotient when N_1 is divided by r, and so on.

$\mathsf{N}_2=er^2+dr+c$, remainder b.

$\mathsf{N}_3=er+d$, remainder c.

$\mathsf{N}_4=e$, remainder d.

Thus the rule is found: **Divide the given number repeatedly by the radix of the scale into which it is to be converted, and**

take the successive remainders in order for the digits, the 1st remainder being the digit in the units place.

The number as originally expressed may be in any scale. It is only necessary to perform the division with due regard to the scale in which it is. An example will make this clear.

(1) Convert 5647 from the scale of 10 to the scale of 8.

(2) Convert 5647 from the scale of 9 to the scale of 8.

The working is as follows:

(1)
$$\begin{array}{r|l} 8 & 5647 \\ 8 & 705\ldots 7 \\ 8 & 88\ldots 1 \\ 8 & 11\ldots 0 \\ & 1\ldots 3 \end{array}$$
Result 13017.

(2)
$$\begin{array}{r|l} 8 & 5647 \\ 8 & 638\ldots 6 \\ 8 & 72\ldots 1 \\ 8 & 8\ldots 1 \\ & 1\ldots 0 \end{array}$$
Result 10116.

In (2) the division is performed thus:

56 in scale $9 = 5 \times 9 + 6 = 51$ in the common scale.

8 into 51 goes 6 times and 3 over.

34 in scale $9 = 3 \times 9 + 4 = 31$ in the common scale.

8 into 31 goes 3 times and 7 over.

77 in scale $9 = 7 \times 9 + 7 = 70$.

8 into 70 goes 8 times and 6 over.

Thus we obtain 638...6.

Each division is worked in the same way.

We may test each of these results by converting the numbers back into their original scales.

(1)
$$\begin{array}{r|l} 10 & 13017 \\ 10 & 1064\ldots 7 \\ 10 & 70\ldots 4 \\ & 5\ldots 6 \end{array}$$
5647.

(2)
$$\begin{array}{r|l} 9 & 10116 \\ 9 & 717\ldots 7 \\ 9 & 63\ldots 4 \\ & 5\ldots 6 \end{array}$$
5647.

Example. Multiply 3651 by 523 in the scale of 7 (septenary scale).

$$\begin{array}{r} 3651 \\ 523 \\ \hline 25545 \\ 10632 \\ 14613 \\ \hline 3612033 \end{array}$$

It will be sufficient to explain the 1st line of the multiplication.

3651×5.

$1 \times 5 = 5$ which is put down.

$5 \times 5 = 25 = 3 \times 7 + 4$. Put down 4 and carry 3.

$6 \times 5 + 3 = 33 = 4 \times 7 + 5$. Put down 5 and carry 4.

$3 \times 5 + 4 = 19 = 2 \times 7 + 5$. Put down 25.

In the 3rd column of the addition $6 + 3 + 5 = 14 = 2 \times 7 + 0$.

In the 4th column 2 (carried) $+ 4 + 6 + 4 = 16 = 2 \times 7 + 2$.

In the 5th column 2 (carried) $+ 1 + 0 + 5 = 8 = 1 \times 7 + 1$.

In the 6th column 1 (carried) $+ 1 + 5 = 7 = 1 \times 7 + 0$.

In the 7th column 1 (carried) $+ 2 = 3$.

294. *To convert a decimal into a radix fraction.*

Let D represent the decimal, and $abcd$ the radix fraction which is in the scale of r.

$$\mathsf{D} = \frac{a}{r} + \frac{b}{r^2} + \frac{c}{r^3} + \frac{d}{r^4}.$$

By multiplying by r we get $a + \frac{b}{r} + \frac{c}{r^2} + \frac{d}{r^3}$, *i.e.* $a + \mathsf{D}_1$.

The integral part a gives us the first required digit.

$$\mathsf{D}_1 r = \left(\frac{b}{r} + \frac{c}{r^2} + \frac{d}{r^3}\right) r = b + \frac{c}{r} + \frac{d}{r^2}.$$

The integral part b gives us the second required digit, and so on until the operation either comes to an end or we get as many digits as are required.

Example. Convert ·7234 from scale 10 into scale 6.

$$\begin{array}{r} \cdot 7234 \\ \underline{\quad\quad 6} \\ 4 \cdot 3404 \\ \underline{\quad\quad 6} \\ 2\ 0424 \\ \underline{\quad\quad 6} \\ 0 \cdot 2544 \\ \underline{\quad\quad 6} \\ \underline{1 \cdot 5264} \end{array} \qquad \text{Result } \cdot 4201 \ldots$$

Example. How may a mass of 236 lbs. be weighed with no weights except 1 lb., 2 lbs., 2^2 lbs., 2^3 lbs., etc., only one of each sort being used. This question can be solved by converting 236 into the scale of 2.

$$\begin{array}{r|l} 2 & 236 \\ 2 & 118 \ldots 0 \\ 2 & 59 \ldots 0 \\ 2 & 29 \ldots 1 \\ 2 & 14 \ldots 1 \\ 2 & 7 \ldots 0 \\ 2 & 3 \ldots 1 \\ & 1 \ldots 1 \end{array}$$

From this it is clear that $236 = 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^3 + 1 \times 2^2$.

$\therefore$ the weights used are 2^7, 2^6, 2^5, 2^3, 2^2.

295. *If a number* N *in the scale of* r *be divided by* $r-1$ *the remainder is the same as that which arises from dividing the sum of the digits of* N *by* $r-1$.

Let the digits of N be a_n, a_{n-1}, $\ldots$ a_1, a_0 being the units digit.

$$\begin{aligned} \mathsf{N} &= a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r + a_0 \\ &= a_n (r^n - 1) + a_{n-1} (r^{n-1} - 1) + \ldots + a_2 (r^2 - 1) + a_1 (r - 1) \\ &\qquad + a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0. \end{aligned}$$

But $r^n - 1$, $r^{n-1} - 1$, etc., are divisible by $r - 1$;

$$\therefore \ \frac{\mathsf{N}}{r-1} = \text{an integer} + \frac{\text{the sum of the digits}}{r-1}.$$

$\therefore$ the only remainder is that which arises from dividing the sum of the digits by $r - 1$.

An important use is made of this in testing the results of multiplication in the common scale.

The remainder when a number is divided by 9 is at once got by adding up the digits of the number and seeing what remainder is found after division by 9.

Thus it is easy to express multiplier and multiplicand in the form $9m + a$ and $9n + b$.

The product is $81mn + 9(an + bm) + ab$.

$\therefore$ the product when divided by 9 gives the same remainder as ab divided by 9.

This testing is called **casting out the nines**.

Example.

437... the 'nine-over' is 5
618 6
2622
427
3496
269966... the 'nine-over' should be 3.

Suppose it is required to test the correctness of this multiplication.

In the multiplicand the remainder after dividing by nine, or the 'nine-over,' is 5; in the multiplier it is 6; therefore in the product it ought to be 30, or rather 3 (since $30 \div 9$ gives remainder 3). It is seen to be 2 on examination of the product; therefore the result is wrong. The error will be found in the 2nd line of multiplication.

Examples. XLVII. a.

Convert the following:

1. 432 from scale 10 to scale 7.
2. 678 from scale 10 to scale 6.
3. 11111 ..... 9.
4. 505 ..... 11.
5. 1739 ..... 12.
6. ·568 ..... 5.
7. ·361 ..... 8.
8. ·95 ..... 6.
9. 23·125 ..... 4.
10. 3050 from scale 6 to scale 10.
11. 100*e* from scale 12 to scale 10.
12. 5324 ..... 7 ..... 9.
13. 647 ..... 8 ..... 6.
14. Multiply 2*t*1 by 38 in scale 11.
15. Convert 2*t*1 and 38 from scale 11 to scale 10, multiply the results together and convert to scale 11.
16. Express the sum of series $2+5\times7+6\times7^2+3\times7^3$ as a number in the scale of 7, and hence find its value in the ordinary scale.
17. The number 11111 is in the scale of 5. Express it in full as a series and find its sum (1) as a G.P., (2) by converting 11111 into the scale of 10.

Examples. XLVII. b.

1. Express 2157 in scale 8.
2. Express 16935 in scale 7.
3. ..... 471664 in scale 12.
4. ..... 62·48 in scale 5, and 16·935 in scale 7.
5. Express ·5 in scale 7, and prove the result by summing a G.P.
6. Transform 101211 from scale 9 to scale 7.
7. ..... 122·2 ..... 4 ..... 10.
8. ..... 26·3 ..... 12 ..... 10.
9. ..... 9294 ..... 12 ..... 9.
10. ..... *t*7*e*9*ee* ..... 12 ..... 10.
11. Multiply 26·3 by 16·7 in scale 12.
12. Divide 252710 by 249 in scale 12. Test the result by reducing dividend, divisor and quotient to scale 10.
13. Find the sq. rt. of 223551 in scale 8; prove by multiplication.

14. Find sq. rt. of 365·738 in scale 9 to 3 places.

15. In what scale is 4072 expressed by 30504?

16. In what scale is 564 expressed by 686?

17. Find a number which is expressed by the same 2 digits whether the scale be 6 or 8.

18. Prove that in any scale 10404 is a square. If the square root be 51 in scale 10, what is the scale in which 10404 is expressed?

19. What number of 2 digits is expressed by the same digits in the scales of 5 and 7?

20. Show how to weigh 85 lbs. using only weights 1, 2, 2^2, 2^3, ... lbs.

21. Weigh 95 lbs. with weights 1, 3, 3^2, ... lbs. Weigh it also using not more than one of each sort of weight, but in either scale.

22. Weigh 304 lbs. using the weights 1, 3, 3^2, ... lbs., not more than one of each, but in either scale.

23. Test the correctness of the following by "casting out the nines":

$674 \times 504 = 339596$. $743 \times 2957 = 2197051$.

$333^2 = 110789$. $37246 \times 117 = 397278$.

$314^2 = 98576$.

What errors will this method not detect?

24. In a scale of radix r the difference between any number and another containing the same digits is divisible by $r - 1$.

CHAPTER XLVIII.

PERMUTATIONS AND COMBINATIONS.

296. A group of r things chosen out of n things without reference to order is called a **combination** or an **r-combination.** The number of them that can be formed is denoted by ${}^nC_r$.

A group of r things chosen out of n things and arranged in a definite order is called a **permutation.** The number of them that can be formed is denoted by ${}^nP_r$.

Out of four letters a, b, c, d we can form the following combinations taking them 3 together: *abc*, *abd*, *acd*, *bcd*.

Each of these groups can be arranged in 6 different ways.

Thus ${}^4C_3 = 4$; but ${}^4P_3 = 24$.

Note that *abc*, *cba*, *cab* all form the same combination but are different permutations.

297. *To find the number of permutations of n different things taken r together.*

Denote the n things by n letters a, b, c, etc.

Choose any letter to stand first. This may be done in n ways. If a be chosen, it may be followed by any one of the remaining $n-1$ letters.

Thus, whichever of the n letters stands first, it will have a choice of $n-1$ letters to follow it.

$$\therefore \ {}^{n}\mathsf{P}_2 = n(n-1).$$

Thus whichever of the $n(n-1)$ pairs stands first, it will have a choice of $n-2$ letters to follow it.

$$\therefore \ {}^{n}\mathsf{P}_3 = n(n-1)(n-2).$$

Continuing this process we see that

$${}^{n}\mathsf{P}_r = n(n-1)(n-2) \ldots \text{continued to } r \text{ factors}$$
$$= n(n-1)(n-2) \ldots (n-r+1).$$

Corollary. ${}^{n}\mathsf{P}_n = n(n-1)(n-2) \ldots 1$. This product (of all integers from 1 to n) is denoted by $\underline{|n}$ or $n!$, and is called **factorial n.**

Examples. XLVIII. a.

1. Put down all the groups of 3 letters which can be formed out of the 4 letters a, b, c, d. How many are there?

2. How many different arrangements of these can be made? Put down all these.

3. Put down all the arrangements of 3 of the 5 letters a, b, c, d, e which can be made with ab standing first in each. How many are there?

4. Put down all the arrangements of the 5 letters a, b, c, d, e such that each begins with a and ends with e, or *vice versa*. How many are there?

5. Arrange *bbbcc* in all possible ways taking them all. How many are there?

6. Simplify ${}^{n}\mathsf{P}_r \div {}^{n-1}\mathsf{P}_r$.

7. How many factors are there in $n \,.\, \overline{n-1} \,.\, \overline{n-2} \ldots \overline{n-6}$?

8. How many factors are there in $n \,.\, \overline{n-1} \,.\, \overline{n-2} \ldots 7$?

9. $n \,.\, \overline{n-1} \,.\, \overline{n-2} \ldots$ to r factors. What is the last one?

10. In ${}^{18}\mathsf{P}_{15} \div \underline{|15}$ how many factors are alike in numerator and denominator. What is the result of cancelling them?

11. If any value can be attached to $\underline{|0}$, what is it? Put $n=1$ in the identity $n\underline{|n-1} = \underline{|n}$.

298. *To find the number of permutations taken all together of n things which are not all different.*

Suppose them to be n letters, of which p letters are a, q letters are b, r letters are c, and the rest are all different.

Let P be the required number of permutations.

If the p letters a were changed into p fresh letters different from the rest, the number of permutations would be thereby multiplied by $\lfloor p$, since the group of p fresh letters would admit of $\lfloor p$ arrangements.

Similarly in the case of the q letters b and the r letters c.

$\therefore$ the total number of permutations would then be

$$\mathsf{P} \cdot \lfloor p \cdot \lfloor q \cdot \lfloor r.$$

But there would then be n letters all different, and consequently the number of permutations of them taken all together would be $\lfloor n$.

$$\therefore \quad \mathsf{P} \cdot \lfloor p \cdot \lfloor q \cdot \lfloor r = \lfloor n;$$

$$\therefore \quad \mathsf{P} = \frac{\lfloor n}{\lfloor p \, \lfloor q \, \lfloor r}.$$

299. *Permutations in a ring.*

In solving questions of this sort it is necessary to observe that permutations which are distinct from each other when arranged in line may cease to be different when arranged in a circle.

Thus *abcdef* and *fabcde* become identical orders if disposed in circular form.

The artifice of fixing one of the persons or things and moving the rest will be found useful.

Notice the difference between arranging persons in a ring and arranging beads to form a necklace.

Examples. XLVIII. b.

1. Find ${}^{7}\mathsf{P}_4$, ${}^{9}\mathsf{P}_3$, ${}^{10}\mathsf{P}_5$.
2. How many changes can be rung on a peal of 7 bells, using all?
3. In how many ways can n persons be arranged in a ring?
4. Four gentlemen and 4 ladies are to be seated alternately at a round table. In how many ways can it be done?
5. If ${}^{n}\mathsf{P}_3 : {}^{n}\mathsf{P}_4 = 1 : 7$, find n.
6. If ${}^{n}\mathsf{P}_4 = 840$, find n.
7. ${}^{n}\mathsf{P}_3 = 20n$; find n.

8. Find the number of arrangements of the letters of *Sirocco, combination, syzygy.*

9. In how many ways can 5 persons be arranged in a ring?

10. In how many ways can 7 different beads be arranged on a string to form a necklace?

11. In how many ways can 6 persons be arranged round a table?

12. In how many ways can a committee of n persons be seated round a table, if the chairman must always be opposite to the fireplace?

300. It should be noticed that, if it is desired to find the number of r-combinations in each of which a specified thing occurs, it is simply necessary to put aside the specified thing in order to join it to each of the combinations $r-1$ together that can be formed out of the other $n-1$ things.

Result ${}^{n-1}\mathsf{C}_{r-1}$.

301. *To find the number of* **r-combinations** *of n things.*

Each combination of r letters produces ${}^{r}\mathsf{P}_{r}$ permutations,

i.e. $\underline{|r}$ permutations.

$$\therefore\ {}^{n}\mathsf{P}_{r} = \underline{|r} \,.\, {}^{n}\mathsf{C}_{r};$$

$$\therefore\ {}^{n}\mathsf{C}_{r} = \frac{{}^{n}\mathsf{P}_{r}}{\underline{|r}} = \frac{n \,.\, \overline{n-1} \,.\, \overline{n-2} \ldots \overline{n-r+1}}{\underline{|r}}$$

$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

302. *To find the number of* **r-combinations** *of n things without assuming the number of permutations.*

Denote the things by a, b, c, etc.

The number of combinations in which a occurs is found by combining the remaining $n-1$ letters $r-1$ together, and is therefore ${}^{n-1}\mathsf{C}_{r-1}$.

Similarly for each of the n letters.

$\therefore$ the total number of r-combinations thus obtained

$$= n \,.\, {}^{n-1}\mathsf{C}_{r-1}.$$

But in these we shall find each combination repeated r times. *e.g. abcd* ... (consisting of r letters) will be found amongst those containing a, amongst those containing b, c, and so on.

$$\therefore\ {}^{n}C_{r}=\frac{n}{r}\cdot{}^{n-1}C_{r-1}=\frac{n}{r}\cdot\frac{n-1}{r-1}\cdot{}^{n-2}C_{r-2}$$

$$=\dots$$

$$=\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\,\dots\,\overline{n-r+2}}{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,\dots\,2}\cdot{}^{n-r+1}C_{1}$$

$$=\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\,\dots\,\overline{n-r+1}}{\underline{\lfloor r}}.$$

Multiply by $\dfrac{\lfloor n-r}{\lfloor n-r}$; $\quad\therefore\ {}^{n}C_{r}=\dfrac{\lfloor n}{\lfloor r\ \lfloor n-r}$.

303. *The number of combinations of n things taken $n-r$ at a time = the number when taken r at a time.*

$${}^{n}C_{n-r}=\frac{\lfloor n}{\lfloor n-r\ \lfloor n-(n-r)}=\frac{\lfloor n}{\lfloor n-r\ \lfloor r}={}^{n}C_{r}.$$

This fact is also evident from the consideration that every choice of r things leaves behind $n-r$ things.

Thus the number of ways of forming a group of r things is the number of ways of forming another group (a *complementary* group) of $n-r$ things,

$$i.e.\ {}^{n}C_{r}={}^{n}C_{n-r}.$$

304. *To prove* ${}^{n}C_{r}+{}^{n}C_{r-1}={}^{n+1}C_{r}$.

${}^{n}C_{r-1}$ is the number of combinations of $n+1$ things r together which contain a specified thing;

${}^{n}C_{r}$ is the number of combinations of $n+1$ things r together which do not contain that specified thing.

The sum of these two must be the total number of combinations of $n+1$ things r together;

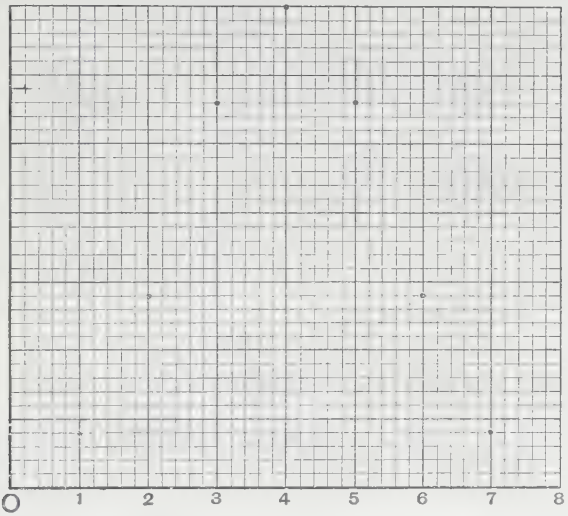
$$\therefore\ {}^{n}C_{r}+{}^{n}C_{r-1}={}^{n+1}C_{r}.$$

As an exercise prove this from the value of ${}^{n}C_{r}$. (Art. 316.)

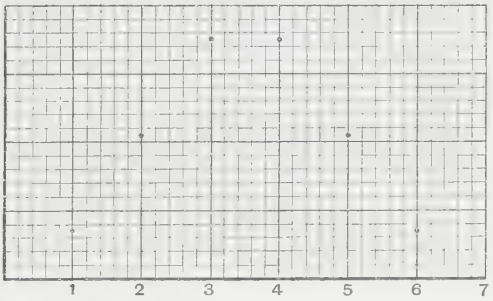
305. *Given n find for what value of r ${}^{n}C_{r}$ is greatest.*

Using r for abscissa and ${}^{n}C_{r}$ for ordinate, we may represent the different values of ${}^{n}C_{r}$ graphically.

Taking n equal to 8, we obtain a symmetrical graph with a single summit at ${}^8C_4$.



Taking n to be 7, we obtain a symmetrical graph with two equally high summits, ${}^7C_3$ and the complementary ${}^7C_4$.



From this it appears that when n is even the maximum value of ${}^nC_r$ is given by $r=\frac{n}{2}$: and when n is odd the maximum is when

$$r=\frac{n-1}{2} \text{ or } \frac{n+1}{2}.$$

306. *To find for what value of r ${}^nC_r$ is greatest, supposing n given.*
The formal proof is as follows:

$$ {}^nC_{r+1} = \frac{n(n-1)\dots(n-r+1)(n-r)}{1\,.\,2\dots r\,.\,(r+1)}; $$

$$ {}^nC_r = \frac{n(n-1)\dots(n-r+1)}{1\,.\,2\dots r}; $$

$$ \therefore \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1}; $$

$$ \therefore {}^nC_{r+1} = \frac{n-r}{r+1}\,{}^nC_r. $$

Now $\frac{n-r}{r+1}$ diminishes as r increases, but is at first > 1.

$$ \therefore {}^nC_{r+1} > {}^nC_r \text{ as long as } \frac{n-r}{r+1} > 1; $$

$\therefore$ ${}^nC_r$ has its greatest value when $\frac{n-r}{r+1} = 1$ or is first < 1.

i.e. when $n-r = r+1$ „ „ $< r+1$.

„ „ $n-1 = 2r$ „ „ $< 2r$.

„ „ $r = \frac{n-1}{2}$ „ „ $> \frac{n-1}{2}$.

(i) If n is odd

the greatest value of ${}^nC_r$ is $ {}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}} = \dfrac{\lfloor n}{\lfloor\frac{n-1}{2}\,\lfloor\frac{n+1}{2}}. $

(ii) If n is even

the greatest value of ${}^nC_r$ is $ {}^nC_{\frac{n}{2}} = \dfrac{\lfloor n}{\lfloor\frac{n}{2}\,\lfloor\frac{n}{2}}. $

307. Questions sometimes occur in which it is required to find the number of ways of dividing up n things into parcels.

The method of solution follows at once from the formula for ${}^nC_r$.

Suppose that it is required to divide n things into two parcels, one containing r things, the other containing the rest.

The number of ways of choosing r things for the first parcel is ${}^nC_r$, and the remaining things form the second parcel.

Thus the number of ways of forming the two parcels is the same as the number of ways of choosing the contents which make up the first parcel.

$$\text{Result} = {}^nC_r = \frac{\lfloor n}{\lfloor r \,\lfloor n-r}.$$

If three parcels are required, we can first divide the whole number into two parcels, and then subdivide one of these.

Example. In how many ways can 10 letters be made up into 2 parcels of 6 and 4 respectively?

The number of ways is $\frac{\lfloor 10}{\lfloor 6 \,\lfloor 4}$, *i.e.* 210.

In how many ways can 10 letters be left at 2 houses, 6 at one, 4 at the other?

The 6 letters can be left at the first house, the 4 at the other, *or vice versa*.

$\therefore$ the number of ways is $\frac{2 \lfloor 10}{\lfloor 6 \,\lfloor 4}$, *i.e.* 420.

308. Interesting applications of the theory of Combinations occur in geometry; *e.g.* the enumeration of the diagonals of a given polygon. The following is an instance of such questions.

Example. Find the number of regions into which a plane is divided by n straight lines, of which no two are parallel and no three meet in a point. (Trin. Coll. Cambridge.)

When there is no straight line, the plane forms one region. The introduction of a straight line divides it into 2 regions.

The addition of a second line makes 4 regions in all,

i.e. the 2nd line adds on 2 regions, making $1+1+2$ regions,

the 3rd ..... adds on 3 regions, making $1+1+2+3$ regions,

and so on.

Thus the total number of regions, when there are n lines,

$$= 1+1+2+3+\ldots+n$$

$$= 1+\frac{n(n+1)}{2} = \frac{1}{2}(n^2+n+2).$$

Examples. XLVIII. c.

1. Find ${}^9C_3$, ${}^9C_6$, ${}^8C_5$, ${}^8C_3$, ${}^{12}C_{10}$.
2. If ${}^{15}C_r = {}^{15}C_6$, find r.
3. If ${}^nP_r = 120\ {}^nC_r$, find r.
4. Find ${}^{10}C_4$ and ${}^{10}C_5$. Add the results and prove that ${}^{11}C_4$ is obtained. State and prove the general theorem of which this is a particular instance.

5. In how many ways can a party of 5 men be chosen from 10 men? In how many ways if (1) one man is always to be included, (2) if one man is always to be excluded?

6. A committee of 7 is to be formed from 10 Conservatives and 8 Liberals, so that there may be 4 Conservatives and 3 Liberals on it. How many different Committees can be formed?

7. Out of 17 consonants and 5 vowels how many words can be formed containing 2 consonants and 1 vowel?

8. How many words of 11 letters may be formed of the letters in $a^3b^5c^2d$?

9. How many words may be formed of all the letters of *accommodation*?

10. How many elevens may be formed out of 15 cricketers, a place being reserved for one particular man?

11. How many straight lines can be made by joining 4 points, no 3 of which are collinear?

12. If $y = {}^8C_x$ plot the positions of the point (x, y) as x takes successive integral values. Find what value of x gives a maximum value to y.

(Take the unit for y much greater than that for x.)

13. In how many ways can a crew of 8 men be arranged in a boat? If only 3 are fit to row stroke, in how many ways can they then be arranged?

14. How many different permutations can be made of the letters of the word *essences*, and how many of these will begin with n and end with s?

15. You have 7 envelopes addressed to 7 people, to 4 of whom you intend to send copies of a circular, the other 3 envelopes to be used for a different purpose. In how many correct ways can you put the 4 circulars into 4 of the 7 envelopes?

16. My cousin belongs to a club of 30 members. In how many ways can I meet at most 5 of the members, of whom my cousin may or may not be one; and in how many ways can I meet 5 members including my cousin?

17. How many numbers can be formed of the digits 1, 2, 3, 4, each digit occurring once in a number? Find the sum of all such numbers.

18. If 5 trees be selected from each of 6 different plantations, in how many ways can they be planted in 6 rows, each consisting of trees from the same plantation?

19. If ${}^nP_3 : {}^nC_5 = 1 : 5$, find n.

20. Out of a dozen persons how many parties can be made, each consisting of not more than 8?

21. From a company of 96 men including 7 officers a picket of 5 has to be chosen. In how many ways can it be done so as to include one and only one officer?

22. A party is chosen from 18 men. The number of ways of choosing it is greater than it would be for a party of a different size. Of how many does it consist? How many of these selections contain 3 given men?

23. Find the number of words of 2 consonants and 2 vowels formed from "education." In how many of these will the two sorts occur alternately?

24. If ${}^{n+1}C_{r+2} = 9 \times {}^nC_{r+1} = 90 \times {}^{n-1}C_r$, find n and r.

25. How many triangles can be formed having their vertices at those of a given pentagon?

26. Fourteen different coins are contained in 2 bags, 7 in each. How many different combinations of 8 coins can be formed by taking 4 from each bag?

27. How many different sets for lawn-tennis can be formed from 10 ladies and 6 gentlemen, each set containing 2 ladies and 2 gentlemen?

28. In how many different ways can 52 cards be dealt into 4 hands of 13 each? In how many different ways can they be dealt to 4 players, 13 to each?

29. Twenty-six people are to travel by an omnibus which can carry 12 inside and 14 outside. If 8 of them will not ride outside and 6 will not go inside, in how many different ways can the party travel?

30. If there are in a bag sixteen balls, in how many different ways may 6 balls be drawn out, so that in each drawing there may be 2 specified balls?

31. The number of combinations of n letters, taken 5 together, in which a, b, c occur is 21. Find the number of combinations of them, taken 6 together, in which a, b, c, d occur.

32. How many different numbers may be composed with the figures in 111223, each consisting of 6 figures?

33. In the decimal system of notation how many numbers are there consisting of 4 digits? Show that the sum of all such numbers is 49495500.

34. I wish to give away 15 different postage-stamps to 3 boys, 5 to each. In how many ways can I do it?

35. Of 15 men, 10 can row and cannot steer, and 5 can steer and cannot row; find how many crews of 8 rowers and a coxswain can be formed out of the 15 men.

36. There are two sets of concurrent straight lines, n in one set, p in the other; how many intersections are there if there are no parallels?

37. In how many ways can 7 sovereigns and 5 shillings be given to 6 men and 6 women, one coin to each?

38. All the numbers made up of the four figures 2, 3, 5, 7 variously arranged are taken. Find the sum of them.

39. Prove ${}^5C_1+{}^5C_2+{}^5C_3+{}^5C_4+{}^5C_5=2^5-1$.

40. Find the number of permutations of the letters of Kassassin taken all together.

41. The number of ways in which n books can be arranged on a shelf so that 2 particular books are not together is $(n-2)\,\underline{|\,n-1}$.

42. How many $\triangle$s are formed by 2 sets of 3 str. lines, each set passing through a point, and no two lines being parallel?

43. How many different numbers of 3 digits can be formed, it being allowed to use the digits 2, 3, 4, 5 once and 6 twice?

44. Find the number of arrangements that can be made of the letters of the word *infinite* (1) when they are taken all together, (2) when they are taken 4 together, so that each arrangement has 2 different vowels and 2 different consonants.

45. A bag contains n sovereigns and n shillings. Prove that the number of ways in which they can be drawn out in succession, one at a time, is $\dfrac{1.3.5\ldots(2n-1)}{\underline{|n}}\times 2^n$.

CHAPTER XLIX.

BINOMIAL THEOREM.

309. By means of the binomial theorem, any binomial expression may be raised to any required power.

The theorem states that

$$(\mathbf{a+b})^n = \mathbf{a}^n + \mathbf{n}\mathbf{a}^{n-1}\mathbf{b} + \frac{\mathbf{n(n-1)}}{\mathbf{1\cdot 2}}\mathbf{a}^{n-2}\mathbf{b}^2 + \frac{\mathbf{n(n-1)(n-2)}}{\mathbf{1\cdot 2\cdot 3}}\mathbf{a}^{n-3}\mathbf{b}^3 + \ldots.$$

The terms on the right hand side are called the expansion of $(a+b)^n$.

The truth of the theorem may easily be tested in simple cases.

Thus $(a+b)^2 = a^2 + 2ab + \frac{2\cdot 1}{1\cdot 2}b^2 = a^2 + 2ab + b^2.$

$$(a+b)^3 = a^3 + 3a^2b + \frac{3\cdot 2}{1\cdot 2}ab^2 + \frac{3\cdot 2\cdot 1}{1\cdot 2\cdot 3}b^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a+b)^4 = a^4 + 4a^3b + \frac{4\cdot 3}{1\cdot 2}a^2b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}ab^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}b^4$$
$$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Note carefully the following rules :

1. The index of the highest power is n, and its coefficient is unity.
2. The number of terms is $n+1$. *E.g.* there are 7 terms in the expansion of $(a+b)^6$.
3. The indices of a decrease by unity, and those of b increase by unity in each successive term.
4. The coefficients of terms equidistant from the beginning and the end of the series are equal.
5. Counting unity as a factor, the number of factors in the numerator of any coefficient is equal to the number of factors in its denominator.
6. The coefficient of the *second* term is ${}^nC_1$, that of the *third* term ${}^nC_2$ and so on.

   Thus the coefficient of the r^{th} term will be ${}^nC_{r-1}$, and that of the $(r+1)^{\text{th}}$ term ${}^nC_r$.
7. The sum of the indices of a and b is n in every term.

In other words, the expansion is homogeneous and of the n^{th} degree.

Writing $-b$ instead of b,

$$(\mathbf{a-b})^n = \mathbf{a}^n + \mathbf{n}\mathbf{a}^{n-1}(-\mathbf{b}) + \frac{\mathbf{n(n-1)}}{\mathbf{1.2}}\mathbf{a}^{n-2}(-\mathbf{b})^2 + \frac{\mathbf{n(n-1)(n-2)}}{\mathbf{1.2.3}}\mathbf{a}^{n-3}(-\mathbf{b})^3 + \dots$$

$$= \mathbf{a}^n - \mathbf{n}\mathbf{a}^{n-1}\mathbf{b} + \frac{\mathbf{n(n-1)}}{\mathbf{1.2}}\mathbf{a}^{n-2}\mathbf{b}^2 - \frac{\mathbf{n(n-1)(n-2)}}{\mathbf{1.2.3}}\mathbf{a}^{n-3}\mathbf{b}^3 + \dots .$$

310. The following is a rapid method of obtaining an expansion in simple cases.

Take $(a+b)^9$.

The first term $= a^9$; the second term $= 9a^8b$;

the third term $= \frac{9.8}{2}a^7b^2 = 36a^7b^2$.

The coefficient of the third term is obtained from that of the second **by multiplying the index of a by the coefficient in the second term, and dividing their product by the number (2) of the term.**

Thus the fourth term $= \frac{7.36}{3}a^6b^3 = 84a^6b^3$.

The fifth term $= \frac{6.84}{4}a^5b^4 = 126a^5b^4$.

The sixth term $= \frac{5.126}{5}a^4b^5 = 126a^4b^5$.

The seventh term $= \frac{4.126}{6}a^3b^6 = 84a^2b^6$, and so on.

Thus $(a+b)^9 = a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$.

311. *Find the general term in the expansion of* $(a+x)^n$.

Taking the general term to be the $(r+1)^{\text{th}}$, as that contains x^r, we see that the coefficient is ${}^n\mathbf{C}_r$.

Also as x occurs with the index r, the index of a must be $n-r$.

$\therefore$ the general term is

$$\frac{n.\overline{n-1}.\overline{n-2}\dots\overline{n-r+1}}{\lfloor r}a^{n-r}x^r.$$

If we had to write down the general term of $(a-x)^n$, we should have to write $-x$ for x;

$\therefore$ the general term of $(a-x)^n$

$$= \frac{n \cdot \overline{n-1} \ldots \overline{n-r+1}}{\lfloor r} a^{n-r}(-x)^r$$

$$= (-1)^r \cdot \frac{n \cdot \overline{n-1} \ldots \overline{n-r+1}}{\lfloor r} a^{n-r}x^r.$$

Using T_r to denote the r^{th} term in a binomial expansion we notice that T_{r+1} in the expansion of $(a+x)^n$ contains x^r.

Also the expansion is homogeneous with respect to a and x, *i.e.* the sum of their indices is the same in all terms and is equal to n.

Thus the term containing x^r must contain a^{n-r}.

$\therefore$ in the expansion of $(a+x)^n$,

$$\mathsf{T}_{r+1} = {}^n\mathrm{C}_r a^{n-r}x^r = \frac{n \cdot \overline{n-1} \ldots \overline{n-r+1}}{\lfloor r} a^{n-r}x^r. \quad \ldots\ldots\ldots\ldots (1)$$

$$\mathsf{T}_r = \frac{n \cdot \overline{n-1} \ldots \overline{n-r+2}}{\lfloor r-1} a^{n-r+1}x^{r-1};$$

$$\therefore \quad \mathsf{T}_{r+1} = \frac{n-r+1}{r} \cdot \frac{x}{a} \cdot \mathsf{T}_r.$$

We can thus derive any term from the preceding term.

This is a general statement of the method employed in Art. 310.

T_{r+1} may be put in the form $\dfrac{\lfloor n}{\lfloor r \; \lfloor n-r} a^{n-r}x^r$ by multiplying numerator and denominator of the coefficient in (1) by $\lfloor n-r$.

312. Example. Expand $(a+x)^2(a-x)^6$.

Expand $(a-x)^6$, writing down only the coefficients.

Multiply by $a+x$ and again by $a+x$ by the method of detached coefficients.

$$\begin{array}{l} 1-6+15-20+15-6+1 \\ 1+1 \\ \hline 1-6+15-20+15-\ 6+1 \\ \quad +1-\ 6+15-20+15-6+1 \\ \hline 1-5+\ 9-\ 5-\ 5+\ 9-5+1 \\ 1+1 \\ \hline 1-5+\ 9-\ 5-\ 5+\ 9-5+1 \\ \quad +1-\ 5+\ 9-\ 5-\ 5+9-5+1 \\ \hline 1-4+\ 4+\ 4-10+\ 4+4-4+1 \end{array}$$

The work might be shortened by observing that each coefficient in the product by $x+a$ is obtained by adding to each coefficient the preceding one.

$$1-6+15-20+15-6+1$$

after the 1st multiplication becomes

$$(1+0)+(-6+1)+(15-6)+(-20+15)+(15-20)+(-6+15)+(1-6)+(0+1)$$

$$1-5+9-5-5+9-5+1,$$

and at the next stage becomes

$$1-4+4+4-10+4+4-4+1.$$

Result $a^8-4a^7x+4a^6x^2+4a^5x^3-10a^4x^4+4a^3x^5+4a^2x^6-4ax^7+x^8$.

313. *To find the sum of the coefficients in the expansion of* $(1+x)^n$.

Write $(1+x)^n$ in the form $c_0+c_1x+c_2x^2+c_3x^3+\ldots+c_nx^n$.

Put $x=1$. Then $c_0+c_1+c_2+\ldots+c_n=(1+1)^n=2^n$. .....(1)

When $x=-1$, we have

$$c_0-c_1+c_2-c_3+\ldots+(-1)^nc_n=(1-1)^n=0^n, \quad \ldots\ldots(2)$$

i.e. the sum of the even coeffs. = sum of odd coeffs.

$$=\frac{2^n}{2} \text{ from } (1)=2^{n-1}.$$

314. *To find* $c_0^2+c_1^2+c_2^2+\ldots+c_n^2$.

$$(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n,$$

$$(x+1)^n=c_0x^n+c_1x^{n-1}+c_2x^{n-2}+\ldots+c_n;$$

$\therefore$ by multiplication $c_0^2+c_1^2+c_2^2+\ldots+c_n^2=$ coeff. of x^n in the product $(1+x)^n(x+1)^n$, *i.e.* in the expansion of $(1+x)^{2n}$;

$$\therefore \quad c_0^2+c_1^2+c_2^2+\ldots+c_n^2=\frac{\lfloor 2n}{\lfloor n \ \lfloor n}.$$

Examples. XLIX. a. (*Chiefly oral.*)

In the expansion of $(a+x)^n$:

1. How many terms are there?
2. Which term contains x^5?
3. What power of a accompanies x^5?
4. What is the coefficient of the 3rd term?
5. ..... 5th term?
6. Why is the $(r+1)^{\text{th}}$ term chosen for the *general* term?
7. What is the denominator of the coefficient of this term?
8. When you have expanded $(a+x)^n$ how can you alter it into $(a-x)^n$?
9. If there are 17 terms, *which* is the middle term?
10. Name in order the coefficients in the expansion of $(1+x)^4$.
11. ..... $(1+x)^5$.
12. ..... $(1+x)^6$.

13. What multiplier will give the $(r+1)^{\text{th}}$ term from the r^{th} in $(1+x)^n$?

14. What multiplier in the case of $(a+x)^n$?

15. Which is the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^6$?

16. What is its value?

17. How is the sum of the coefficients of $(1+x)^n$ found?

18. $(a+x)^n = a^n + (\text{terms containing a factor } n) + x^n$. If in this we put $n=0$ and deduce that $(a+x)^0 = a^0 + 0 + x^0$, *i.e.* $1 = 1+1$, where is our mistake?

19. In the expansion of $(\sqrt{3}+\sqrt{2})^6$ which terms are rational?

20. In $(a+x)^{12}$ which term has the same coefficient as the 5^{th}?

21. .....10^{th}?

22. In $(a+x)^n$ one term contains a^3x^7. What is n?

How many factors are there in the product

23. $n(n-1)(n-2)\ldots(n-9)$?

24. $n(n-1)(n-2)\ldots(n-p)$?

25. $n(n-1)(n-2)\ldots(n-r+1)$?

26. $n(n-1)(n-2)\ldots(n-r+2)$?

27. $n(n-1)(n-2)\ldots(n-r-1)$?

28. In the expansion of $(a+x)^n$, what number from the beginning is:

(i) the third term from the end? (ii) the fifth term from the end?

(iii) the r^{th}.....? (iv) the $\overline{r+1}^{\text{th}}$.....?

29. In the expansion of $(a+x)^{n-1}$, what number from the beginning is:

(i) the fourth term from the end?

(ii) the seventh.....?

(iii) the r^{th}.....?

(iv) the $(r+1)^{\text{th}}$.....?

30. (i) Which is the middle term in the expansion of $(x+a)^{2n}$?

(ii) Write down its value.

(iii) Write down the term immediately preceding the middle term.

(iv) .....following .....

31. Write down the last four terms in the expansion of $(a+x)^n$ in ascending powers of x.

32. (i) Write down the r^{th} term in the expansion of $(x+z)^{25}$.

(ii) .....$\overline{r+1}^{\text{th}}$.....

(iii) .....$\overline{r-1}^{\text{th}}$.....

(iv) .....r^{th} term from the end.....

(v) .....$\overline{r+1}^{\text{th}}$.....

315. *To prove the Binomial Theorem for a positive integral index.*

$$(x+a)^n = (x+a)(x+a)(x+a)\ldots n \text{ factors.}$$

The product of these is a homogeneous expression of the n^{th} degree.

A term containing a^r contains also x^{n-r}.

What is its coefficient? This term is found from the above product by selecting a from r of the n factors and x from each of the $n-r$ remaining factors.

The a can be selected in ${}^nC_r$ ways.

$\therefore$ the term $a^r x^{n-r}$ appears multiplied by ${}^nC_r$. Similarly for the other terms.

$$\therefore\ (x+a)^n = x^n + {}^nC_1 a x^{n-1} + {}^nC_2 a^2 x^{n-2} + \ldots + a^n,$$

and the number of terms in the expansion is $n+1$.

316. *To prove the Binomial Theorem for a positive integral index.* (*2nd method.*)

Assume it true when the index is n,

i.e. $(x+a)^n = x^n + {}^nC_1 a x^{n-1} + {}^nC_2 a^2 x^{n-2} + \ldots + {}^nC_r a^r x^{n-r} + \ldots$.

Multiply both sides by $x+a$;

$$\begin{aligned}\therefore\ (x+a)^{n+1} &= x^{n+1} + {}^nC_1 a x^n + {}^nC_2 a^2 x^{n-1} + \ldots + {}^nC_r a^r x^{n-r+1} + \ldots \\ &\quad + a x^n + {}^nC_1 a^2 x^{n-1} + \ldots + {}^nC_{r-1} a^r x^{n-r+1} + \ldots \\ &= x^{n+1} + ({}^nC_1 + 1) a x^n + ({}^nC_2 + {}^nC_1) a^2 x^{n-1} + \ldots \\ &\quad + ({}^nC_r + {}^nC_{r-1}) a^r x^{n-r+1} + \ldots.\end{aligned}$$

But ${}^nC_r + {}^nC_{r-1}$

$$= \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \ldots \overline{n-r+1}}{\lfloor r} + \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \ldots \overline{n-r+2}}{\lfloor r-1}$$

$$= \frac{n \cdot \overline{n-1} \ldots \overline{n-r+2}}{\lfloor r-1} \left\{ \frac{n-r+1}{r} + 1 \right\}$$

$$= \frac{n \cdot \overline{n-1} \ldots \overline{n-r+2}}{\underline{r-1}} \cdot \frac{n+1}{r}$$

$$= \frac{\overline{n+1} \cdot n \cdot \overline{n-1} \ldots \overline{n-r+2}}{\lfloor r} = {}^{n+1}C_r.$$

Similarly ${}^nC_1 + 1 = {}^{n+1}C_1$, ${}^nC_2 + {}^nC_1 = {}^{n+1}C_2$, etc.;

$\therefore\ (x+a)^{n+1} = x^{n+1} + {}^{n+1}C_1 a x^n + {}^{n+1}C_2 a^2 x^{n-1} + \ldots + {}^{n+1}C_r a^r x^{n-r+1} + \ldots$.

Thus, if the theorem hold for index n, it holds for $n+1$.

But it does hold for index 2; $\therefore$ for index 3; $\therefore$ for index 4, and so on.

$\therefore$ the theorem is true for all positive integral values of the index.

317. *To find the greatest term in the expansion of* $(1+x)^n$, *n being a positive integer.*

$$\mathsf{T}_{r+1} = \frac{n-r+1}{r} x \,.\, \mathsf{T}_r = \left(\frac{n+1}{r} - 1\right) x \,.\, \mathsf{T}_r ;$$

$\therefore$ T_r is greatest, or equal to the greatest term,

when $\left(\frac{n+1}{r} - 1\right) x$ is first $<$ or $= 1$,

i.e. ..... $\frac{(n+1)x}{r}$ ..... $x+1$,

..... $(n+1)x$ ..... $r(x+1)$,

..... r ..... $>$ or $= \frac{(n+1)x}{x+1}$.

(1) Let $\frac{(n+1)x}{x+1}$ be an integer, say p.

Then T_{r+1} is greater than T_r as long as $r < p$; (for instance when $r = p-1$ or $p-2$ or any number below p);

i.e. $\mathsf{T}_p >$ any term before it.

When $r = p$ the multiplier becomes 1;

$$\therefore \quad \mathsf{T}_{p+1} = \mathsf{T}_p.$$

Thus the terms increase till we get to the p^{th}, the next term is equal to this, and after that they decrease;

$\therefore$ $\mathsf{T}_p, \mathsf{T}_{p+1}$ are the greatest terms.

(2) Let $\frac{(n+1)x}{x+1}$ be a fraction. Let its integral part be q.

Here the multiplier > 1 as long as $r < q+$ a fraction,

i.e. the terms increase.

For values of r from $q+1$ onwards the terms decrease.

Thus $\mathsf{T}_{q+1} >$ any term before,

and $\mathsf{T}_{q+1} > \mathsf{T}_q$ (for the multiplier > 1);

$\therefore$ T_{q+1} is the greatest term.

The greatest term of $(a+x)^n$ may be found by putting it in the form $a^n\left(1+\frac{x}{a}\right)^n$. For in this case we have only to find which

is the greatest term in $\left(1+\frac{x}{a}\right)^n$; and this is done by using $\frac{x}{a}$ instead of x in this article.

Examples. XLIX. b.

Expand

1. $(1+x)^5$.
2. $(a+x)^7$.
3. $(a^2-x^2)^6$.
4. $\left(ab-\frac{x}{2}\right)^8$.
5. $\left(x+\frac{1}{x}\right)^8$.
6. $\left(ax-\frac{b}{x}\right)^7$.
7. $(3x+2y)^5$.
8. $(a+b\sqrt{2})^8+(a-b\sqrt{2})^8$.
9. Give the 5th term in the expansion of $(1-x)^{10}$.
10. .....6th..... .....$(a^2+x^2)^{11}$.
11. ..... 12th.....$(3-y)^{15}$.
12. Give the middle term in the expansion of $\left(x+\frac{1}{x}\right)^{12}$.
13. Give the two middle terms in the expansion of $\left(2x-\frac{1}{x}\right)^{11}$.
14. n being a positive integer, find T_{r+1} in the expansion of $(1+x)^{2n}$ and T_{r+1} in the expansion of $(3-2x)^n$.
15. A = the sum of the odd terms, B the sum of the even terms in the expansion of $(a+x)^n$; prove that $\mathsf{A}^2-\mathsf{B}^2=(a^2-x^2)^n$.
16. The coefficient of x^n in $(1+x)^{2n}=\frac{\lfloor 2n}{\lfloor n \lfloor n}=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{\lfloor n}2^n$.
17. In the expansion of $(1+x)^n$ given $x^2\cdot \mathsf{T}_{r+1}=\mathsf{T}_{r+3}$, find r.

Write down the general term in

18. $(1+2x)^n$.
19. $(3-5x)^{2n}$.
20. $(a^2-3bx)^n$.
21. $\left(x+\frac{1}{x}\right)^n$.
22. Write down the middle term of $\left(x^2-\frac{1}{x^2}\right)^{2n}$.
23. Write down the two middle terms of $(2x+y)^9$.
24. Find the coefficient of x^7 in $(2x+x^2)^5$.
25. Find the coefficient of x^8 in $\left(x^2+\frac{1}{x}\right)^7$.
26. Find the term independent of x in $\left(x+\frac{2}{x}\right)^{10}$.
27. Show that $(a+\sqrt{2})^n+(a-\sqrt{2})^n$ contains no surds.
28. Expand $(a-x)^7(a+x)^2$ by detached coefficients.
29. Expand $\left(x-\frac{1}{x}\right)^5\left(x+\frac{1}{x}\right)^3$.....
30. Simplify $(\sqrt{3}+1)^8-(\sqrt{3}-1)^8$.

31. Prove $\left(x+\frac{1}{x}\right)^6-\left(x-\frac{1}{x}\right)^6=12\left(x^4+\frac{1}{x^4}+\frac{10}{3}\right)$.

32. Show by actual multiplication that the whole coefficient of x^4 in the product of the expansions of $(1+x)^n$ and $(1-x)^n$ = the coefficient of x^4 in the expansion of $(1-x^2)^n$.

33. Find the ratio of the middle term to the one before it in the expansion of $(1-2x)^{20}$.

34. If a_r denote the coefficient of x^r in $(1-x)^{2n-1}$, prove that
$$a_{r-1}+a_{2n-r}=0.$$

35. Expand $(1+a_1)(1+a_2)(1+a_3)(1+a_4)$, and show how $(1-a)^4$ may be derived from the result.

If $a=\cdot0003$, find, correct to 4 decimal places, $(1-a)^4$.

36. Find the greatest coefficient in the expansion of $(1+x)^{14}$.

37. Find the greatest term in the expansion of $(\frac{1}{2}+x)^8$ where $x=1$.

38. ... . $(3+2b)^9$ where $b=2$.

39. ..... $(4+9x)^6$ where $x=1$.

40. ..... $(1+x)^6$ where $x=\frac{1}{2}$.

41. ..... $(\frac{1}{2}+\frac{1}{3}x)^7$ where $x=1$.

42. Find the sum of the coefficients in the expansion of $(2x+3y)^n$.

43. ..... $(5a-4b)^n$.

44. Simplify $1+{}^nC_1\cdot\frac{x}{1+x}+{}^nC_2\left(\frac{x}{1+x}\right)^2+\ldots+\left(\frac{x}{1+x}\right)^n$.

45. Prove $\left(x+\frac{1}{x}\right)^{20}=x^{20}+\frac{1}{x^{20}}+{}^{20}C_1\left(x^{18}+\frac{1}{x^{18}}\right)+{}^{20}C_2\left(x^{16}+\frac{1}{x^{16}}\right)+\ldots$
and give the last term.

46. The coefficients of the 5th and 7th terms of $(1+x)^n$ are 70 and 28; find n.

47. Find the coefficient of x^9 in $(1-2x)^3(1+x)^8$.

When n is a positive integer let the expansion of $(1+x)^n$ be written as
$$c_0+c_1x+c_2x^2+c_3x^3+\ldots+c_nx^n.$$

Prove the following:

48. $c_1+c_3+c_5+\ldots=2^{n-1}$.

49. $c_1+2c_2+3c_3+\ldots+nc_n=n\,.\,2^{n-1}$.

50. $c_0+\frac{1}{2}c_1+\frac{1}{3}c_2+\ldots\frac{c_n}{n+1}=\frac{2^{n+1}-1}{n+1}$.

51. $c_0c_1+c_1c_2+c_2c_3+\ldots+c_{n-1}c_n=\frac{\lfloor 2n}{\lfloor n-1\ \lfloor n+1}$.

[Coefficient of x^{n+1} in $(1+x)^n(x+1)^n$.]

52. $c_0c_r+c_1c_{r+1}+c_2c_{r+2}+\ldots+c_{n-r}c_n=\frac{\lfloor 2n}{\underline{n-r}\ \underline{n+r}}$.

53. $c_0-\frac{1}{2}c_1+\frac{1}{3}c_2-\ldots+(-1)^n\frac{c_n}{n+1}=\frac{1}{n+1}$.

54. The sum of the products of c_0, c_1, c_2, etc.... taken two at a time
$$=2^{2n-1}-\frac{\lfloor 2n}{2\lfloor n\ \lfloor n}.$$

55. Sum the series $1+{}^nC_2x^2+{}^nC_4x^4+\ldots$.

318. *The Binomial Theorem when the exponent (or index) is not a positive integer.*

The Theorem has been proved to be true if the index be any positive integer. It remains to examine its truth for *any* value of n, where the expression $(1+x)^n$ has to be expanded.

When n is a positive integer the expansion terminates; for the coefficients contain factors n, $n-1$, $n-2$, etc., and we reach a coefficient which vanishes and thus the end of the series is reached.

When n is negative or fractional the series of terms goes on for ever, and it will be found that the expansion is not intelligible *unless x is less than unity.* The proof of the necessity for this restriction is beyond our scope at this stage, but it will be assumed in all expansions of $(1+x)^n$ where n is negative or fractional that $x<1$.

If $f(n)$ denotes $1+nx+\frac{n\,.\,\overline{n-1}}{\underline{2}}x^2+\frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{\underline{3}}x^3+\dots$,

the following should be written out for practice:

$$f(7),\ f(8),\ f(-2),\ f(-3),\ f(\tfrac{1}{2}),\ f(m+n).$$

It has been proved that $f(8)$ is the same as $(1+x)^8$.

It remains to be proved that $f(\frac{1}{2})$ is the same as $(1+x)^{\frac{1}{2}}$, that $f(\frac{1}{3})$ is $(1+x)^{\frac{1}{3}}$, etc., and that $f(-n)$ is $(1+x)^{-n}$, *when* $x<1$.

319. *To prove the Binomial Theorem for any negative index.*

$$\{1+2x+3x^2+4x^3+\dots \text{ to infinity}\}(1-x)$$
$$=1+2x+3x^2+4x^3+5x^4+\dots-x-2x^2-3x^3-4x^4-\dots$$
$$=1+x+x^2+x^3+x^4+\dots \text{ to infinity}$$
$$=\text{an infinite G.P.}$$
$$=\frac{1}{1-x},\ \textit{if } x \textit{ be numerically less than } 1,$$
$$=(1-x)^{-1};$$
$$\therefore\ 1+2x+3x^2+4x^3+\dots \text{ to infinity}=(1-x)^{-2}.$$

Substituting $-x$ for x we get

$$1-2x+3x^2-4x^3+\dots \text{ to infinity}=(1+x)^{-2},$$

i.e. $f(-2)=(1+x)^{-2}$.

Again

$$\left\{1+(n+1)x+\frac{(n+1)(n+2)}{\underline{|2}}x^2+\frac{(n+1)(n+2)(n+3)}{\underline{|3}}x^3+\ldots\right\}(1-x)$$

$$=1+(\overline{n+1}-1)x+\left(\frac{\overline{n+1}\,.\,\overline{n+2}}{\underline{|2}}-\overline{n+1}\right)x^2$$

$$+\left(\frac{\overline{n+1}\,.\,\overline{n+2}\,.\,\overline{n+3}}{\underline{|3}}-\frac{\overline{n+1}\,.\,\overline{n+2}}{\underline{|2}}\right)x^3+\ldots$$

$$=1+nx+\frac{n\,.\,\overline{n+1}}{\underline{|2}}x^2+\frac{n\,.\,\overline{n+1}\,.\,\overline{n+2}}{\underline{|3}}x^3+\ldots.$$

Putting $-x$ for x we get

$$\left\{1-\overline{n+1}\,.\,x+\frac{\overline{n+1}\,.\,\overline{n+2}}{\underline{|2}}x^2-\frac{\overline{n+1}\,.\,\overline{n+2}\,.\,\overline{n+3}}{\underline{|3}}x^3+\ldots\right\}(1+x)$$

$$=1-nx+\frac{n\,.\,\overline{n+1}}{\underline{|2}}x^2-\frac{n\,.\,\overline{n+1}\,.\,\overline{n+2}}{\underline{|3}}x^3+\ldots,$$

i.e. $f(-\overline{n+1})=f(-n)\,.\,(1+x)^{-1}$;

$\therefore$ if $f(-n)=(1+x)^{-n}$, we see that $f(-\overline{n+1})=(1+x)^{-\overline{n+1}}$.

But we know that $f(-1)=(1+x)^{-1}$ *if* $x<1$,

and $f(-2)=(1+x)^{-2}$;

$\therefore$ $f(-3)=(1+x)^{-3}$ by induction.

Hence $f(-4)=(1+x)^{-4}$, and so on;

$\therefore$ for all integral values of n, if $x<1$, we have

$$(1+x)^{-n}=f(-n)=1-nx+\frac{n\,.\,\overline{n+1}}{\underline{|2}}x^2-\frac{n\,.\,\overline{n+1}\,.\,\overline{n+2}}{\underline{|3}}x^3+\ldots.$$

Thus the Binomial Theorem is proved for *any* integral index.

320. *To prove the Binomial Theorem for any fractional index.*

Let $f(m)=1+mx+\dfrac{m\,.\,\overline{m-1}}{\underline{|2}}x^2+\dfrac{m\,.\,\overline{m-1}\,.\,\overline{m-2}}{\underline{|3}}x^3+\ldots.$

Then $f(n)=1+nx+\dfrac{n\,.\,\overline{n-1}}{\underline{|2}}x^2+\dfrac{n\,.\,\overline{n-1}\,.\,\overline{n-2}}{\underline{|3}}x^3+\ldots.$

By actual multiplication of these two series we can prove that certainly as far as a few terms

$$f(m)\times f(n)=1+(m+n)x+\frac{\overline{m+n}\,.\,\overline{m+n-1}}{\underline{|2}}x^2+\ldots.$$

If x were greater than 1, the multiplication of these two series which have an unlimited number of terms, would become unintelligible; for they contain terms which would become infinite. If however $x<1$, we may assume that, as it is mere multiplication, the *form* of the result is the same whatever letters are employed in it, and whatever be the values of those letters.

$\therefore$ to aid our unfinished multiplication we may perform it with m and n treated as positive integers, and say that the result will hold when they are changed to fractional or negative quantities.

Now if m and n were positive integers $f(m)$ and $f(n)$ would be $(1+x)^m$ and $(1+x)^n$.

$\therefore$ $f(m)\times f(n)$ would be $(1+x)^m\times(1+x^n)$, *i.e.* $(1+x)^{m+n}$, *i.e.* $f(m+n)$.

$\therefore$ whatever m and n may be $f(m)\times f(n)=f(m+n)$. .....(1)

$\therefore$ $f(m)\times f(n)\times f(p)=f(m+n)\times f(p)=f(m+n+p)$;

and so on.

$\therefore$ $f(m)\times f(n)\times f(p)\times \ldots=f(m+n+p+\ldots)$. .....(2)

In (2) let $m=n=p=\ldots=\frac{r}{s}$ where r and s are positive integers, and let there be s of these quantities.

$$\therefore\ f\left(\frac{r}{s}\right)\times f\left(\frac{r}{s}\right)\times f\left(\frac{r}{s}\right)\times\ldots=f\left(\frac{r}{s}+\frac{r}{s}+\frac{r}{s}+\ldots \text{ to } s \text{ terms}\right);$$

$$\therefore\ f\left(\frac{r}{s}\right)^s=f(r)=(1+x)^r \text{ since } r \text{ is a positive integer};$$

$$\therefore\ (1+x)^{\frac{r}{s}}=f\left(\frac{r}{s}\right)=1+\frac{r}{s}x+\frac{\frac{r}{s}\cdot\left(\frac{r}{s}-1\right)}{\lfloor 2}x^2+\ldots,$$

which is the Binomial Theorem with a fractional exponent.

To examine the case of a negative exponent.

In (1) put $-m$ for n;

$$\therefore\ f(m)\times f(-m)=f(m-m)=f(0)=1.$$

$$\therefore\ f(-m)=\frac{1}{f(m)}=\frac{1}{(1+x)^m}=(1+x)^{-m};$$

$$\therefore\ (1+x)^{-m}=f(-m)$$
$$=1+(-m)x+\frac{(-m)(-m-1)}{\lfloor 2}x^2+\ldots.$$

Example 1. Expand $(1+x)^{-\frac{2}{3}}$ and find the $(r+1)^{\text{th}}$ term.

$$(1+x)^{-\frac{2}{3}}=1+(-\tfrac{2}{3})x+\frac{(-\frac{2}{3})(-\frac{5}{3})}{1\cdot 2}x^2+\frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{1\cdot 2\cdot 3}x^3+\dots.$$

$$\mathsf{T}_{r+1}=\frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})\dots(-\frac{2}{3}-r+1)}{1\cdot 2\cdot 3\dots r}x^r$$

$$=(-1)^r\cdot\frac{2\cdot 5\cdot 8\dots(3r-1)}{3^r\cdot\underline{|r}}x^r.$$

Example 2. Expand $(1+x)^{-4}$ and find the $(r+1)^{\text{th}}$ term.

$$(1+x)^{-4}=1+(-4)x+\frac{(-4)(-5)}{1\cdot 2}x^2+\frac{(-4)(-5)(-6)}{1\cdot 2\cdot 3}x^3+\dots.$$

$$\mathsf{T}_{r+1}=\frac{(-4)(-5)(-6)\dots(-4-r+1)}{1\cdot 2\cdot 3\dots r}x^r$$

$$=(-1)^r\cdot\frac{4\cdot 5\cdot 6\dots r\cdot(r+1)(r+2)(r+3)}{1\cdot 2\cdot 3\cdot 4\cdot 5\dots r}x^r$$

$$=(-1)^r\cdot\frac{(r+1)(r+2)(r+3)}{1\cdot 2\cdot 3}x^r.$$

Notice that $(1-\mathbf{x})^{-n}$ has all its terms positive.

Also that $(1-\mathbf{x})^{-1}=1+\mathbf{x}+\mathbf{x}^2+\mathbf{x}^3+\dots.$

$$(1-\mathbf{x})^{-2}=1+2\mathbf{x}+3\mathbf{x}^2+\dots+(\mathbf{r}+1)\mathbf{x}^r+\dots,$$

$$(1-\mathbf{x})^{-3}=1+3\mathbf{x}+\frac{3\cdot 4}{1\cdot 2}\mathbf{x}^2+\frac{4\cdot 5}{1\cdot 2}\mathbf{x}^3+\dots$$

$$+\frac{(\mathbf{r}+1)(\mathbf{r}+2)}{1\cdot 2}\mathbf{x}^r+\dots.$$

321. *To find the greatest term without regard to sign in the expansion of $(a\pm x)^{\pm n}$ in ascending powers of x.*

The $(r+1)^{\text{th}}$ term is obtained from the r^{th} (without regard to sign) by the multiplier

$$\frac{n-r+1}{r}\cdot\frac{x}{a}\text{ (in the case of a positive index),}$$

$$\text{or }\frac{n+r-1}{r}\cdot\frac{x}{a}\text{ (in the case of a negative index).}$$

$\therefore\ \mathsf{T}_{r+1}>\mathsf{T}_r$ as long as this multiplier >1.

Now since the multiplier may be put in the form $\left(\frac{n+1}{r}-1\right)\frac{x}{a}$ or $\left(\frac{n-1}{r}+1\right)\frac{x}{a}$, it continually decreases as r increases.

The r^{th} term is greatest or equal to the greatest when the multiplier first $<$ or $= 1$,

i.e. (1) $\dfrac{n+1}{r}x <$ or $= a+x$, or (2) $\dfrac{n-1}{r}x <$ or $= a-x$.

Thus the r^{th} term is the greatest or $=$ the greatest

when r is first $>$ or $= \dfrac{(n+1)x}{a+x}$ (with the positive index),

or ..... $>$ or $= \dfrac{(n-1)x}{a-x}$ (with the negative index).

Example 1. Find the greatest term in the expansion of $(1+x)^{\frac{11}{2}}$ where $x=\frac{2}{3}$.

$$\mathsf{T}_{r+1} = \mathsf{T}_r \cdot \frac{n-r+1}{r} \cdot x = \mathsf{T}_r \cdot \frac{\frac{11}{2}-r+1}{r} \cdot \frac{2}{3} = \mathsf{T}_r \cdot \frac{13-2r}{3r}.$$

$\therefore$ T_r is greatest, or equal to the greatest term,

when $\dfrac{13-2r}{3r}$ is first $<$ or $= 1$,

i.e. ..... $13-2r$ ..... $= 3r$,

i.e. ..... $5r$ ..... $>$ or $= 13$.

The smallest integer greater than $\frac{13}{5}$ is 3.

$\therefore$ the *third* term is the greatest.

Example 2. Find the greatest term in the expansion of $(1-x)^{-5}$ where $x=\frac{3}{4}$.

Here T_{r+1} is numerically equal to $\mathsf{T}_r \cdot \dfrac{5+r-1}{r} \cdot \dfrac{3}{4}$

$$= \mathsf{T}_r \cdot \frac{4+r}{r} \cdot \frac{3}{4}$$

$$= \mathsf{T}_r \cdot \frac{12+3r}{4r}.$$

$\therefore$ T_r is greatest, or equal to the greatest term,

when $\dfrac{12+3r}{4r}$ is first $<$ or $= 1$,

i.e. ..... $12+3r$ ..... $4r$,

i.e. ..... r ..... $>$ or $= 12$.

If $r=12$, the multiplier $= 1$;

$\therefore$ the 12^{th} and 13^{th} terms are equal and are the greatest.

Example 3. Find the greatest coefficient in the expansion of $(4x+5y)^n$.

By putting $x=y=1$ we reduce the terms to their coefficients.

$\therefore$ it is only necessary to find the greatest *term* in the expansion of $(4+5)^n$.

Examples. XLIX. c.

Expand to 4 terms

1. $(a-2x)^{-5}$.
2. $(1+2x)^{\frac{1}{2}}$.
3. $(1-x)^{-\frac{1}{2}}$.
4. $(1+\frac{2}{3}x)^{-5}$.
5. $(a^2-x^2)^{\frac{1}{2}}$.
6. $(1+\frac{1}{2}x)^{\frac{1}{2}}$.
7. $(1-x^2)^{-\frac{1}{3}}$.
8. $(1-3x)^{-\frac{1}{3}}$.
9. $\dfrac{1}{a^2+b^2}$.
10. Find the 4th term of $(1+x)^{\frac{1}{2}}$.
11. ..... 5th ..... $(3a-2b)^{-10}$.
12. Find the coefficient of x^{12} in $(1-b^3x^2)^{\frac{5}{2}}$.
13. Find the $(r+1)^{\text{th}}$ term of $(x+a)^{-n}$.
14. ..... $(1-x)^{-3}$.
15. ..... $(1-x)^{-5}$.
16. ..... $(2-x)^{-5}$.
17. ..... $(1+x)^{-\frac{2}{3}}$.
18. The coefficient of x^r in $(1-4x)^{-\frac{1}{2}}$ is $\dfrac{\lfloor 2r}{(\lfloor r)^2}$.
19. Expand to 4 terms $\left(1-\dfrac{1}{x}\right)^{-n}$. Write down the $(n+1)^{\text{th}}$ term and find its value when $x=\dfrac{1}{n}$.
20. Find the coefficient of x^r in $\left(x+\dfrac{1}{x}\right)^n$.
21. Find the coefficient of x^{10} in the expansion of $(x+x^2)^{-3}$.
22. ..... x^r .... $\dfrac{(3+2x)}{(1-x)^2}$.
23. ... . x^r ..... $\dfrac{(1-3x)^2}{(1-x)^3}$.
24. ..... x^5 ..... $\dfrac{1-2x+3x^2}{(1-x)^4}$.
25. ..... x^{2r} ..... $\dfrac{1-x}{(1+x)^2}$.
26. ..... x^r .... $\dfrac{3-x}{(1-x)^3}$.
27. The coefficient of x^r in the expansion of $(1+x+x^2+\ldots+x^{n-1})^n$
$= -n +$ coefficient of x^n in $(1+x)^{2n-1}$.
28. Find the coefficient of x^r in $(1+2x+3x^2+\ldots \textit{ad inf.})^{-n}$.
29. Find the first negative term of $(1+x)^{\frac{5}{2}}$.
30. How many terms of $(1-3x)^{\frac{7}{2}}$ are positive?
31. Which is the greatest term of $(1+x)^{\frac{5}{2}}$ when $x=\frac{2}{3}$.
32. .... $(1-x)^{-8}$ when $x=\frac{1}{2}$.
33. .... $(1+2x)^{\frac{7}{2}}$ when $x=\frac{2}{5}$.
34. ..... $(1+\frac{3}{4})^{\frac{13}{3}}$.

322. *Find the sum of the first $n+1$ coefficients in the expansion of $(1-x)^{-3}$.*

$$(1-x)^{-3} = 1 + 3x + \frac{3 \cdot 4}{1 \cdot 2}x^2 + \frac{4 \cdot 5}{1 \cdot 2}x^3 + \ldots + \frac{\overline{n+1} \cdot \overline{n+2}}{1 \cdot 2}x^n + \ldots .$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots .$$

Now, in multiplying together the two series on the right, we see that the coefficient of x^n is $1 + 3 + \frac{3 \cdot 4}{1 \cdot 2} + \ldots + \frac{\overline{n+1} \cdot \overline{n+2}}{1 \cdot 2}$;

$\therefore$ the required sum = coefficient of x^n in $(1-x)^{-4}$

$$= \frac{\overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3}}{1 \cdot 2 \cdot 3}.$$

APPROXIMATIONS.

323. If α is a small fraction, α^2 is a small fraction of α.

Thus if α is *very* small, α^2 may be neglected in comparison with α in approximate calculations, or α^3 in comparison with α^2.

If we require a rough approximation we can reject all powers of α above the 1st; if a closer approximation is required we can retain the term containing α^2, and reject the higher powers: and so on according to the degree of accuracy required.

$$(1+\alpha)^2 = 1 + 2\alpha + \alpha^2 ;$$

$\therefore$ if an approximate value of $(1+\alpha)^2$ is required, we may call it $1 + 2\alpha$.

$$(1+\alpha)^3 = 1 + 3\alpha + 3\alpha^2 + \alpha^3 ;$$

$\therefore$ $1 + 3\alpha$ is an approximation for $(1+\alpha)^3$.

A closer approximation is $1 + 3\alpha + 3\alpha^2$.

In the same way,

$$(1+\alpha)^n = 1 + n\alpha + \frac{n \cdot \overline{n-1}}{1 \cdot 2}\alpha^2 + \ldots$$

$= 1 + n\alpha$ approximately, when α is a small fraction.

Also when α and β are both small fractions,

$$(1+\alpha)^m(1+\beta)^n = (1+m\alpha)(1+n\beta) \text{ approx.}$$

$= 1 + m\alpha + n\beta$ approx., neglecting the product $mn\alpha\beta$, which is small compared with α and β.

Example 1. Calculate approximately $\sqrt[4]{623}$.

We know that $625 = 5^4$.

$$\text{Now } \sqrt[4]{623} = (625-2)^{\frac{1}{4}} = 625^{\frac{1}{4}}\left(1-\tfrac{2}{625}\right)^{\frac{1}{4}}$$

$$= 5\left\{1 - \frac{1}{4}\cdot\frac{2}{625} + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{1\,.\,2}\left(\frac{2}{625}\right)^2 - \ldots\right\}$$

$$= 5\{1 - \tfrac{1}{4}(\cdot0032) - \tfrac{3}{32}(\cdot0032)^2 - \ldots\}$$

$$= 5\{1 - \cdot0008 - \cdot00000096\} \text{ approximately}$$

$$= 5\{1 - \cdot00080096\} = 4\cdot9959952.$$

Observing the 3rd term we notice that if we took only 2 terms the accuracy of the result to the first 4 places of decimals would not be affected. The result would then be 4·996.

Example 2. Find the value correct to 9 places of decimals of

$$(1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}} \text{ when } x = \cdot0003.$$

$$(1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}} = (1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \ldots)(1 + x + 2x^2 + \tfrac{14}{3}x^3 + \ldots)$$

$$= 1 + 2x + \tfrac{5}{2}x^2 + \ldots$$

$$= 1 + \cdot0006 + \tfrac{5}{2} \times \cdot00000009 + \tfrac{20}{3} \times \cdot000000000027 + \ldots$$

$$= 1\cdot000600225.$$

This is correct to 9 places, since the 4th term, if it were included, would not alter any of the first 9 places of decimals.

Example 3. If x be small compared with unity, find an approximate value of $\dfrac{(1+x)^{\frac{2}{3}}(1-2x)^{\frac{1}{2}}}{1-\frac{5}{6}x}$.

The expression $= \dfrac{(1+\frac{2}{3}x)(1-x)}{1-\frac{5}{6}x}$ if we expand the factors of the numerator by the Binomial Theorem and neglect x^2 and higher powers of x.

$\therefore$, by multiplication of the factors of the numerator,

$$\text{the expression} = (1-\tfrac{1}{3}x)(1-\tfrac{5}{6}x)^{-1}$$

$$= (1-\tfrac{1}{3}x)(1+\tfrac{5}{6}x)$$

$$= 1 + \frac{x}{2} \text{ approximately.}$$

Example 4. If x be small compared with unity,

$$\frac{(2+3x)^{\frac{1}{2}}(8+5x)^{\frac{2}{3}}}{(1-x)^{\frac{1}{4}} + (4-3x)^{\frac{1}{2}}} = \frac{2^{\frac{1}{2}}(1+\frac{3}{2}x)^{\frac{1}{2}}\,.\,8^{\frac{2}{3}}(1+\frac{5}{8}x)^{\frac{2}{3}}}{1 - \frac{1}{4}x + 4^{\frac{1}{2}}(1-\frac{3}{4}x)^{\frac{1}{2}}}$$

$$= \frac{4\sqrt{2}\,.\,(1+\frac{3}{4}x)(1+\frac{5}{12}x)}{1 - \frac{1}{4}x + 2(1-\frac{3}{8}x)}$$

$$= \frac{4\sqrt{2}\,.\,(1+\frac{7}{6}x)}{3-x} = \frac{4\sqrt{2}(1+\frac{7}{6}x)}{3\left(1-\frac{x}{3}\right)}$$

$$= \tfrac{4}{3}\sqrt{2}(1+\tfrac{7}{6}x)\left(1-\frac{x}{3}\right)^{-1}$$
$$= \tfrac{4}{3}\sqrt{2}(1+\tfrac{7}{6}x)(1+\tfrac{1}{3}x)$$
$$= \tfrac{4}{3}\sqrt{2}(1+\tfrac{3}{2}x) \text{ approximately.}$$

A closer approximation would be obtained by neglecting only powers of x above the 2nd.

Example 5. If l feet be the length of a pendulum which beats seconds, how many beats will it lose in 24 hours if it be lengthened one per cent, the time of a beat in seconds being given by the expression $\pi\sqrt{\frac{l}{32}}$?

Let t be the time (in seconds) of the beat of the lengthened pendulum.

Then $t = \pi\sqrt{\frac{l}{32}\cdot\frac{101}{100}}$.

But $1 = \pi\sqrt{\frac{l}{32}}$ (since the original pendulum beats seconds).

$\therefore$ by division $t = \sqrt{1+\frac{1}{100}}$.

The number of beats in 24 hours = the number of seconds in 24 hrs. $\div$ the time of 1 beat $= \dfrac{24\times 60\times 60}{(1+\frac{1}{100})^{\frac{1}{2}}} = 86400(1+\frac{1}{100})^{-\frac{1}{2}}$

$= 86400(1-\frac{1}{2}\cdot\frac{1}{100})$ approximately

$= 86400(1-\frac{1}{200}) = 86400 - 432.$

It originally did make 86400 beats;

$\therefore$ the number of beats lost = 432.

324. *Approximate division.*

To divide by 997, divide by 1000 and add on 3 thousandths of the result.

$$5758 \div 997 = 5{\cdot}758 + {\cdot}017 = 5{\cdot}775,$$

a result which is correct to 2 places of decimals.

The reason for the method is evident when the working is shown by the Binomial Theorem.

$$5758 \div 997 = 5758 \div (1000-3)$$
$$= 5758 \div 1000(1-\tfrac{3}{1000}) = 5{\cdot}758(1-\tfrac{3}{1000})^{-1}$$
$$= 5{\cdot}758\times(1+\tfrac{3}{1000}) \text{ approximately}$$
$$= 5{\cdot}758 + \tfrac{3}{1000} \text{ of } 5{\cdot}758.$$

In the same way we might divide by 1003; but in this case the addition would be replaced by subtraction.

325. Approximation for the r^{th} root of a number.

If $\sqrt[r]{\mathsf{N}} = a + x$ where a is an approximation and consequently x is small, then a closer approximation is given by the formula

$$\sqrt[r]{\mathsf{N}} = \frac{(r+1)\mathsf{N} + (r-1)a^r}{(r-1)\mathsf{N} + (r+1)a^r} \cdot a \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

For instance in finding cube root

$$\sqrt[3]{\mathsf{N}} = \frac{4\mathsf{N} + 2a^3}{2\mathsf{N} + 4a^3} \cdot a = \frac{2\mathsf{N} + a^3}{\mathsf{N} + 2a^3} \cdot a.$$

To find $\sqrt[3]{2}$.

Here an approximation is $\frac{5}{4}$.

$$\therefore \quad \sqrt[3]{2} = \frac{2 \times 2 + \frac{125}{64}}{2 + \frac{250}{64}} \cdot \frac{5}{4} = \frac{190}{152} = 1{\cdot}25992\ldots.$$

This approximation gives the first 5 decimal places of $\sqrt[3]{2}$ correctly.

If $p - q$ be small compared with p or q,

$$\sqrt[r]{\frac{p}{q}} = \frac{(r+1)p + (r-1)q}{(r-1)p + (r+1)q} \text{ approximately.}$$

For $\sqrt[r]{\dfrac{p}{q}} = \dfrac{\{(p+q) + (p-q)\}^{\frac{1}{r}}}{\{(p+q) - (p-q)\}^{\frac{1}{r}}}$

$$= \frac{\{1 + (p-q)/(p+q)\}^{\frac{1}{r}}}{\{1 - (p-q)/(p+q)\}^{\frac{1}{r}}} = \frac{1 + \frac{1}{r}(p-q)/(p+q)}{1 - \frac{1}{r}(p-q)/(p+q)} \text{ nearly}$$

$$= \frac{(r+1)p + (r-1)q}{(r-1)p + (r+1)q}.$$

This becomes the formula given above, if $p = \mathsf{N}$, and $q = a^r$.

Examples. XLIX. d.

1. Find the values of the following:

(i) $(1{\cdot}05)^2$ correct to two decimal places.

(ii) $(1{\cdot}05)^3$ .....

(iii) $(1{\cdot}07)^3$ .....

(iv) $(1\cdot002)^3$ correct to three decimal places.
(v) $(1\cdot04)^4$ ..... .
(vi) $(1\cdot002)^6$ ..... five ..... .
(vii) $(\cdot999)^3$ ..... .
(viii) $(\cdot98)^4$ ..... four ..... .

2. Write down the value of $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$ when $x=10$; and find its value, to the nearest hundredth, when $x=10\cdot1$.

3. Find the value, to the nearest hundredth, of $\left(x+\frac{1}{x}\right)^4$ when $x=10$. Also find the value of the same expression, to the nearest hundredth, when $x=10\cdot1$.

4. Find the value of $\left(x-\frac{1}{x}\right)^3$ when $x=10$. Also find its value, to the nearest hundredth, when $x=9\cdot9$.

5. Test the following rules for approximation:
(i) To divide by π, multiply by $\frac{1}{3}-\frac{1}{100}-\frac{1}{2000}$.
$\left(\text{Compare this expression with the value of } \frac{1}{\pi}.\right)$
(ii) To divide by π^2, multiply by $\frac{1}{10}+\cdot0013$.

6. (i) Find the value of $1\cdot0001^5$ correct to 4 places of decimals.
(ii) ..... $1\cdot0006^{\frac{1}{3}}$ ..... 5 places.
(iii) ..... $\frac{1}{(1-\frac{1}{8}x)^2}$ ..... 8 places when $x=\cdot001$.
(iv) ..... $(1+x)^3(1-x)^{-2}$ ..... 3 places when $x=\cdot0002$.

7. (i) ..... $\sqrt[3]{1005}$ ..... 3 places.
(ii) ..... $\sqrt{99}$ ..... 5 places.
(iii) ....

8. (i) ..... $\sqrt[3]{510}$ ..... .
(ii) ..... $\sqrt[3]{28}$ ..... .
(iii) ..... $\sqrt[5]{3123}$ ..... .

9. The side of a square is increased by $\cdot1$ per cent. What percentage of increase is there in the area?

10. When x is <1, prove that $1-x$ is an approximate value of $\frac{1}{1+x}$, and that $1+x$ is an approximate value of $\frac{1}{1-x}$. What is the amount of the error when $x=\frac{1}{100}$?

11. Find the sq. rt. of $1+x$ as far as the term in x^2. $[x<1.]$

12. Approximate to $\sqrt{1\cdot03}$.

13. If we use the formula $(1-x)^{\frac{1}{2}}=1-\frac{1}{2}x$ to find $\sqrt{99\cdot7}$, to how many decimal places is it correct?

14. Express $\frac{1}{5x+1}$ in ascending and also in descending powers of x.

15. Draw the graphs of $y=\frac{1}{1-x}$, $y=1$, $y=1+x$, $y=1+x+x^2$, on the same diagram.

16. Divide 4831 by 996 approximately.

17. The edge of a cube is diminished by ·01 per cent. What percentage of the original volume is the decrease of volume?

18. A pendulum which beats seconds is lengthened by ·04 per cent. of its length. How many beats does it lose in 24 hours?

$\left[\text{Time of 1 beat}=\pi\sqrt{\frac{l}{32}}\right]$

19. If x is so small that powers above x^2 may be neglected, find the value of $(1+7x)^8-(1+8x)^7$.

When x is very small

20. $\frac{(1-x)^{\frac{1}{2}}+(1+x)^{\frac{2}{3}}}{(1-x)+(1-x)^{\frac{1}{2}}}=1+\frac{5x}{6}$ approximately.

21. $\frac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{4}}}{(1-x)^{\frac{1}{2}}+(1+x)^{\frac{1}{3}}}=1+\frac{5x}{24}$ approximately

22. $\frac{\left(9\frac{x}{2}+1\right)^{\frac{1}{2}}(1-3x^2)^{\frac{1}{3}}}{(1+\frac{9}{8}x)^2}=1-\frac{3\,0\,7}{6\,4}x^2$ approximately.

23. Prove that

$$\frac{\sqrt{3}}{2}=1-\frac{1}{2}\cdot\frac{1}{4}-\frac{1.1}{2.4}\cdot\frac{1}{4^2}-\frac{1.1.3}{2.4.6}\cdot\frac{1}{4^3}-\frac{1.1.3.5}{2.4.6.8}\cdot\frac{1}{4^4}-\ldots$$

24. Prove that $\frac{\sqrt{3}}{\sqrt{2}}=1+\frac{1}{6}+\frac{1.3}{1.2}\cdot\frac{1}{6^2}+\frac{1.3.5}{1.2.3}\cdot\frac{1}{6^3}+\ldots$.

25. Prove that the last series

$$=1+\frac{1}{2}\cdot\frac{1}{2}-\frac{1.1}{2.4}\cdot\frac{1}{2^2}+\frac{1.1.3}{2.4.6}\cdot\frac{1}{2^3}-\frac{1.1.3.5}{2.4.6.8}\cdot\frac{1}{2^4}+\ldots.$$

$\left[\text{Since } (1-\tfrac{1}{3})^{-\frac{1}{2}}=\sqrt{1+\tfrac{1}{2}}.\right]$

Sum the series

26. $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\frac{5}{3^4}+\ldots\ldots$ to infinity.

27. $1+3\cdot\frac{2}{5}+\frac{3.4}{1.2}\cdot\left(\frac{2}{5}\right)^2+\frac{4.5}{1.2}\cdot\left(\frac{2}{5}\right)^3+\ldots.$

28. Prove that $\frac{3\sqrt{2}}{4}=1+\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1.3}{2.4}\cdot\frac{1}{3^4}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3^6}+\ldots.$

Find the sum of the first n coefficients in the expansion of

29. $(1-x)^{-4}$. 30. $(1-x)^{\frac{1}{2}}$. 31. $(1-x)^{-\frac{3}{2}}$. 32. $\frac{1+x}{(1-x)^2}$.

33. Expand $(1-4x)^{\frac{1}{2}}$ to five terms. Show that the coefficient of x^n may be written $-\frac{2\lfloor 2n-2}{\lfloor n\ \ \underline{n-1}}$.

34. Write in its simplest form the $(r+1)^{th}$ term of $\left(x^2-\frac{2}{x}\right)^{-\frac{1}{2}}$.

35. Expand $\left(1-\frac{a}{\mathsf{D}}\right)^{-1}$ in powers of a to the term in a^2. When a balance is used in air of density a, and a body of density D is balanced by it by brass weights of density d weighing m grams, the real weight of the body in grams is $m\left(1-\frac{a}{d}\right)\Big/\left(1-\frac{a}{\mathsf{D}}\right)$.

Expand this expression to the first power of a, and find how much the weight of the body differs from that of the brass weights when $m=1000$, $a=\cdot0012$, $d=8$, $\mathsf{D}=\cdot5$.

36. Find the coefficient of x^n in the expansion of $\frac{1}{1-x-x^2+x^3}$.

37. If $\mathsf{N}=(a+x)^3$ where x is small in comparison with a,

$$\sqrt[3]{\mathsf{N}}=\frac{2\mathsf{N}+a^3}{\mathsf{N}+2a^3}\cdot a \text{ approximately.}$$

CHAPTER L.

REVISION PAPERS.

L. a.

1. Simplify $\frac{x^3(y-z)+y^3(z-x)+z^3(x-y)}{x^2(y-z)+y^2(z-x)+z^2(x-y)}$.

2. Simplify $\frac{2+\sqrt{3}}{4+\sqrt{3}}+\frac{4-2\sqrt{3}}{5-2\sqrt{3}}$.

3. If α, β be the roots of $x^2+px+q=0$, find the equation whose roots are $m\alpha^2+n\beta^2$ and $m\beta^2+n\alpha^2$.

4. Solve $x^2+5x-4+\sqrt{x^2+5x+16}=0$.

5. Prove, without assuming the value of C_r, that $\mathsf{C}_r=\frac{n}{r}\mathsf{C}_{r-1}$.

6. Given $\log 2$, find $\log 312{\cdot}5$, $\log\sqrt{\cdot05}$, $\log \cdot0625^6$.

7. Draw the graphs of $y=2x+\frac{1}{2}$ and $y=4x$ and by them find the simultaneous equations.

L. b.

1. Find square root of $8+2\sqrt{15}$.

2. In how many ways can a crew of 8 be arranged in a boat, if only 3 are fit to row the stroke oar?

3. A man walking from A to B at the rate of $3\frac{1}{2}$ miles an hour starts 40 minutes before the departure of a coach from A that goes 10 miles an hour, and is picked up by it on the way. When he arrives at B, he finds that his journey by coach has lasted 2[illegible]. Find the distance from A to B.

4. If $\log_4 N = \cdot 35184$, find $\log_8 N$.

5. The first and last of an odd number of quantities in H.P. are $\frac{3}{2}-\sqrt{2}$ and $\frac{3}{2}+\sqrt{2}$: find the middle term.

6. Find the conditions which must be satisfied in order that the equations $x^2+ax+b^2=0$, $x^2-bx+a^2=0$, may have (1) the same roots, (2) one root common.

7. If $a:b=b:c$, each ratio $=\dfrac{b^2}{a+b}:\dfrac{c^2}{b+c}$.

L. c.

1. The roots of the equation $ax^2+2bx+c=0$ will be imaginary if a, b, c are in H.P. and have the same sign.

2. Between two numbers whose sum is $2\frac{1}{8}$ an even number of arithmetic means is inserted. The sum of these means exceeds their number by 1. How many means are there?

3. Ten men are to be selected from 4 companies, a different number being taken from each company, and no company being without a representative. How many different arrangements can be made as to the different numbers to be taken from the several companies?

4. If one-tenth of the trees in a plantation are cut down every year, after how many years will less than one-third of the original number be left? [$\log 3 = \cdot 4771$.]

5. Find the 10^{th} term in the expansion of $\left(x-\dfrac{1}{x}\right)^{15}$.

6. Find $\sqrt[5]{\cdot 002}$, using logarithms.

7. The natural numbers are divided into groups thus:

1; 2, 3, 4; 5, 6, 7, 8, 9; etc.

Prove that the sum of the numbers in the n^{th} group is $n^3+(n-1)^3$.

L. d.

1. Find the value of $\dfrac{x+2+\sqrt{4x+x^2}}{x+2-\sqrt{4x+x^2}}$ where $x^{\frac{1}{2}}=a^{\frac{1}{2}}-a^{-\frac{1}{2}}$.

2. Given $\log_{10}5 = \cdot 69897$, find to 3 places $\log\sqrt{\cdot 08}$ to the base 125.

3. 864 bricks each containing 110·592 cub. inches, are piled in a stack, whose length and breadth are each double of its height. What is the height?

4. Find the value of $\sqrt{(3+8\sqrt{7+4\sqrt{3}})}-\sqrt{3}$.

5. Find the square root of $a^{-2}+2a^{-1}(2-b^{-2})+b^{-4}+4(1-b^{-2})$; and prove that it is possible to find the square root of $a+b^{\frac{1}{2}}$ if a^2-b is a perfect square.

6. Find the general term $(r+1)^{\text{th}}$ in the expansion of $\left(3-\dfrac{x}{3}\right)^{-3}$.

7. Find $\sqrt[5]{99999\cdot 5}$ correct to 11 decimal places.

L. e.

1. Solve $x^2-3\sqrt{2x^2-x-2}=x(1-x)$.

2. What number of 2 digits is that which, if its digits be reversed, becomes 1 less than its half, the sum of the digits being 7?

3. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, each of these fractions $=\left(\frac{a^4+c^4+e^4}{b^4+d^4+f^4}\right)^{\frac{1}{4}}=\left(\frac{ace}{bdf}\right)^{\frac{1}{3}}$.

4. Transform 4569·246 from the ordinary scale to the scale of 5, obtaining a correct result to 4 places of fifths.

5. Prove that the greatest number of combinations that can be formed with $2n$ things, each combination containing the same number of things, is double the greatest number that can be formed with $2n-1$ things.

6. Find the 5th term in the expansion of $\left(2x-\frac{x^2}{2}\right)^{\frac{3}{2}}$ and the coeff. of x^r in the expansion of $(1-x)^{-\frac{1}{2}}$.

7. Find to the nearest integer the number of years in which a sum of money doubles itself at 4 per cent. per annum compound interest.

[$\log 2=\cdot3010$, $\log 13=1\cdot1139$.]

L. f.

1. Find an expression, of the 2nd degree in x, which is equal to 2, -1, 34, when $x=1$, -2, 3.

2. Express $2(a^2+b^2+c^2-bc-ca-ab)$ as the sum of three squares.

3. Find by logarithms (i) $\sqrt[7]{2\cdot718}$, (ii) $\cdot296\times3\cdot378$, (iii) e^{π} where $e=2\cdot718$ and $\pi=3\cdot14$.

4. If a, b, c are in G.P., and $a-b$, $c-a$, $b-c$ in H.P., prove that $a+4b+c=0$.

5. Show that $-1^2+2^2-3^2+4^2-\ldots$ to $2n$ terms $=1+2+3+\ldots+2n$.

6. The square of the sum of two numbers is divided by the product of the sum of their cubes and the difference of their squares. Express the result algebraically in lowest terms.

7. A parent puts in a box for a child on every birthday a half-crown for every year of its age. How old will the child be when the whole of the money in the box is £17?

L. g.

1. Solve $\left.\begin{aligned}3x^2+5xy&=22\\11xy-3y^2&=19\end{aligned}\right\}$.

2. Trace the values of $x+\frac{1}{x}$ as x changes from -3 to $+3$, and represent it graphically. Find its least numerical value.

3. Sum the infinite series $\frac{1}{1\cdot05}+\frac{1}{1\cdot05^2}+\frac{1}{1\cdot05^3}+\ldots$.

4. The sum of the squares of the first n natural numbers is $20n$. Find n.

5. How many sums can be made with the following coins: a penny, a sixpence, a shilling, a half-crown, a crown, a sovereign?

6. Represent graphically $11+6\sqrt{2}$. Find its square root graphically and test your accuracy by finding the result algebraically also.

7. If a square number ends in 9, the preceding digit is even.

L. h.

1. A series whose 1st, 2nd, and 3rd terms are

$$\frac{1}{\sqrt{2}}, \quad \frac{1}{1+\sqrt{2}}, \quad \frac{1}{4+3\sqrt{2}}$$

is either Arithmetic or Geometric. Determine which it is, and write down the 4th term.

2. Find the sq. rt. of $3(x-1)+2\sqrt{2x^2-7x-4}$.

3. Find the value of x from the equation $18^{8-4x}=(54\sqrt{2})^{3x-2}$ using as base $3\sqrt{2}$.

4. Prove that, if two numbers have the same digits, their difference is a multiple of 9.

5. In how many ways can 10 persons be divided into parties of 3 and 7?

In how many ways can 10 things be divided between 2 persons so that one may have 3, and the other 7?

6. Find the greatest coefficient in the expansion of

(i) $(1+x)^{21}$, (ii) $(1+x)^{24}$.

7. Find the coefficient of x^r in $(1+2x)/(1-x)^3$.

L. k.

1. A diagonal of a cyclic quadrilateral is given by the expression

$$\left\{\frac{(bc+ad)(ac+bd)}{ab+cd}\right\}^{\frac{1}{2}}$$

where a, b, c, d are the sides taken in order. If the sides, in order, are 60, 25, 52, 39, prove that the diagonals are 65 and 63, and the area is 1764. [Divide into 2 triangles.]

2. Divide $\frac{a}{b}+\frac{b}{a}+2$ by $\left(\frac{a}{b}\right)^{\frac{1}{3}}+\left(\frac{b}{a}\right)^{\frac{1}{3}}-1$.

3. Solve $\sqrt{1+x}+\sqrt{1-x}=\sqrt{3-x}$.

4. What fraction in the common scale is equal to $\cdot 5\dot{3}$ in the scale of 7?

5. If a, b, c are in H.P., then

$$\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)+\frac{3}{b^2}=\frac{4}{ac}.$$

6. Determine which is the greatest term in the expansion of $(3+5x)^8$ when $x=\frac{1}{2}$.

7. Show how to sum the series $1+2x+3x^2+\ldots+nx^{n-1}$; prove that

$$1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2+\ldots+n\left(1+\frac{1}{n}\right)^{n-1}=n^2.$$

L. l.

1. If one of the equations $x^2-x(3c-b)+bc=0$ and $x^2-x(5c-b)+4c^2=0$ has equal roots, so has the other.

2. If 1, x, y are in A.P., and 1, y, x in G.P., find x and y.

3. How many words, each of 7 letters, can be formed from 3 vowels and 4 consonants, in which no two consonants are together?

4. Find log ·00132874 to 6 places, given that

$$\log 1{\cdot}3287 = {\cdot}123427, \quad \log 1{\cdot}3288 = {\cdot}123460.$$

5. The diameter of the earth being 7900 miles and that of the moon 2160, find the diameter of a sphere whose surface is the sum of their surfaces. (Surface $\propto$ square of diameter.)

6. If $(1+x)^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$, show that

$$2c_0 + 2^2\frac{c_1}{2} + 2^3\frac{c_2}{3} + \ldots = \frac{3^{n+1}-1}{n+1}.$$

7. Each of two bags contains 12 different coins; find how many different combinations of 10 coins can be made by taking 5 out of each bag.

L. m.

1. If a, b, c, d are in G.P., prove that

$$(a^2+ac+c^2)(b^2+bd+d^2) = (ab+bc+cd)^2.$$

2. What number added to the expression

$$\frac{x^2}{9} + \frac{1}{4x^2} + \frac{2}{3}x + \frac{1}{x}$$

will make it a square?

3. Sum to infinity $\dfrac{1}{\sqrt{2}} + \dfrac{1}{1+\sqrt{2}} + \dfrac{\sqrt{2}}{3+2\sqrt{2}} + \ldots$.

4. In the equation $x^2 - ax + \dfrac{a^2-b^2}{4} = 0$ find the value of b when the roots are equal.

5. A walks m miles in n hours; B walks $6n$ miles in $\frac{1}{3}m$ hours; and the difference of their rates of walking is $\frac{1}{2}$ mile per hour. Find the rate at which each walks.

6. Show that $(x^2-bc)(2x-b-c)^{-1}$ has no real value between b and c.

7. Solve the equation $a^x(a^x-1) = 2$; and find the value of

$$10 \log \tfrac{3}{2} + 7 \log \tfrac{5}{18} + 4 \log \tfrac{48}{25}.$$

L. n.

1. The product of any 4 consecutive even integers increased by 16 is a perfect square.

2. If $x = 2+\sqrt{2}$, then $x^2 + \dfrac{4}{x^2} = 12$.

3. Find the relation between a, b, c when $x+y=a$, $x^2+y^2=b$, $x^3+y^3=c$.

4. Solve $\left.\begin{aligned} 2(x^4+y^4)+7xy(x^2+y^2) &= 740 \\ 2(x^2+y^2)-xy &= 20 \end{aligned}\right\}$.

5. Ten men are chosen in every possible way out of 16. In how many of the groups do 2 particular men occur?

6. Find the coefficient of x^r in $(1+x)(1-x)^{-4}$.

7. Find which is the greatest term in the expansion of $(1-x)^{-\frac{4}{3}}$ when $x = \frac{1}{11}$.

L. p.

1. If $x=3-\sqrt{3}$, then $x^2+\frac{36}{x^2}=24$.

2. If a, b, c are in A.P., and a, b, $c+1$ are in G.P., prove that
$$a=(a-b)^2=(b-c)^2.$$

3. A number consists of 3 digits in G.P. The sum of the right-hand and left-hand digits exceeds twice the middle digit by 1; and the sum of the left-hand and middle digits is two-thirds of the sum of the middle and right-hand digits. Find the number.

4. Prove ${}^{2n+4}C_{n+2}+{}^{2n+4}C_{n+3}={}^{2n+5}C_{n+3}$.

5. Sum the series $1+2\times6+3\times6^2+4\times6^3+5\times6^4$, by expressing it first in the scale of 6 and then converting it.

6. $105^x=100$. Find x to 4 places.

7. Expand $(1-x)^{-4}$ to 5 terms, and write down the $(r+4)^{\text{th}}$ term in its simplest form.

Apply the Binomial Theorem to prove that
$$1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots \textit{ad inf.}=\sqrt{8}.$$

L. q.

1. A merchant at the end of the 1st year had doubled his original capital, the 2nd year he gained £80 more than the square root of his increased capital, the 3rd year he cleared half the square of all that he had at the end of the 2nd, and found himself with £18240. How much had he at first?

2. The common difference of an A.P. is $-d$, and the sum of n terms is $(2a+d)^2$; find n.
$9d$

3. Prove that $2^{\frac{1}{3}}=2^{\frac{1}{4}}.2^{\frac{1}{16}}.2^{\frac{1}{64}}\ldots$ *ad inf.*, and hence show that it is possible to find cube root by continued extractions of square root and multiplication. Find an approximate value of $2^{\frac{1}{3}}$ by 2 multiplications, and show to how many figures it is correct.

4. Prove that $2^{4n}-1$ is divisible by 15.

5. The equation $y=mx+c$ when combined with $y^2=4ax$ gives equal roots for x and equal roots for y. Find c.

6. By the Binomial Theorem sum the series
$$1+\frac{1}{2^2}+\frac{1.3}{\lfloor 2}\cdot\frac{1}{2^4}+\frac{1.3.5}{\lfloor 3}\cdot\frac{1}{2^6}+\ldots \textit{ad inf.}$$

7. Find the coeff. of x^r in $(1+2x+3x^2+4x^3+\ldots \textit{ad inf.})^n$.

L. r.

1. Simplify $7\log\frac{16}{15}+5\log\frac{25}{24}+3\log\frac{81}{80}$.

2. If a, b, c are in H.P. $a^2+c^2>2b^2$.

3. If $x+y+z=a$, and $x^2+y^2+z^2=b^2$, express $x^3+y^3+z^3-3xyz$ in terms of a and b.

4. Solve the equation $1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\ldots$ *ad inf.* $=\sqrt{1\cdot 25}$.

5. The difference between a number and that number reversed is divisible by 9.

6. Find 5 numbers in A.P. whose sum shall be 25 and their product 2520.

7. In how many ways can a party of 6 be selected from 20 people, so that the party may never contain more than one of two specified?

L. s.

1. If $f(n)=6n^2+5n+1$, find $\sum_1^n\{f(n)\}$.

2. Find the number of ways of dealing 52 cards into 4 "hands," 13 to each. (Give the answer in factorial form.)

3. Find geometrically, the mean proportional to $7+2\sqrt{6}$ and 1, and so obtain graphically $\sqrt{7+2\sqrt{6}}$.

4. Obtain $\sqrt[3]{510}$ approximately by the Binomial Theorem and check your result by logarithms.

5. On a division in the House of Commons, if the number of members for the motion had been increased by 50 from the other side, the motion would have been carried by 5 to 3; but if those against the motion had received 60 from the other party, the motion would have been lost by 4 to 3. Did the motion succeed, and how many members voted?

6. Three consecutive coefficients in the expansion of $(1+x)^n$ are 91, 364 and 1001. Find n.

7. Find the sum of the first r coefficients in the expansion of $(1+x)^5$.

L. t.

1. How many numbers less than 1000 are there which contain the digit 3 at least once?

2. If x is very small $\dfrac{\sqrt{1+x}+(1-x)^{\frac{1}{4}}}{\sqrt{1-x}+(1+x)^{\frac{1}{3}}}=1+\frac{5}{24}x$ nearly.

3. How may a body of 754 lbs. be weighed with weights of 1, 3, 3^2, 3^3 ... lbs., only one of each being used?

4. Draw the graph of $y^2=12x$, and find where it is met by the line $y=2x+\frac{3}{2}$.

5. Prove that $3^{4n}-1$ is divisible by 80.

6. By logarithms calculate the square root of the reciprocal of 6241.

7. If $f(x)=\dfrac{a^x-a^{-x}}{a^x+a^{-x}}$, prove $f(x+y)=\dfrac{f(x)+f(y)}{1+f(x)f(y)}$.

CHAPTER LI.

INTEREST AND ANNUITIES.

326. Simple Interest. If r be the interest on £1 for 1 year, $\mathsf{P}r$ is the interest on P pounds for 1 year.

$\therefore$ the interest on P pounds for n years is $\mathsf{P}nr$.

If V be the present value of P due n years hence, V with its interest must amount to P in n years.

$$\therefore\ \mathsf{V}+\mathsf{V}nr=\mathsf{P};\quad \therefore\ \mathsf{V}=\frac{\mathsf{P}}{1+nr}.$$

$$\text{Discount}=\mathsf{P}-\mathsf{V}=\mathsf{P}-\frac{\mathsf{P}}{1+nr}=\frac{\mathsf{P}nr}{1+nr}.$$

This is called *true discount*. In actual practice the discount allowed is *simple interest*.

327. Compound Interest. Let R be the amount of £1 in 1 year, so that $\mathsf{R}=1+r$.

In 1 year P amounts to PR.

In the 2nd year this PR amounts to $\mathsf{PR}.\mathsf{R}$.

$\therefore$ in 2 years P amounts to PR^2.

In 3 years P amounts to PR^3; and so on.

$\therefore$ the amount of P in n years at compound interest is PR^n.

Let V be the present value of P due n years hence, reckoning compound interest.

V with its interest must amount to P in n years.

$$\therefore\ \mathsf{V}\mathsf{R}^n=\mathsf{P};\quad \therefore\ \mathsf{V}=\mathsf{P}.\mathsf{R}^{-n}.$$

LI. a.

1. At what rate per cent. simple interest will £91. 13*s*. 4*d*. amount to £100 in 4 yrs.?

2. In what time will £315 become £378 at 4 p.c. simple interest?

3. Find the true discount on £720 due 3 months hence at $2\frac{1}{2}$ per cent. simple interest.

4. Find the compound interest on £620 for 5 years at 3 p.c.

5. ..... £50 ..... 10 ..... 4 ... .

6. Find at compound interest the amount of £825 in 6 yrs. at $3\frac{1}{2}$ p.c.

7. ..... £120. 17*s*. 6*d*. ....7 ..... 4 p.c.

8. ..... £30. 5*s*. 10*d*. ...14 ..... 5 p.c.

9. In what time does a sum of money double itself at 5 p.c. compound interest?

10. In what time at 4 p.c.?

11. Find the amount of £10 in 40 years at 5 p.c. compound interest.

12. If the birth rate in a place be 76 in 1000, the death rate 48 in a thousand, in how many years will the population be doubled?

13. **At simple interest, the interest minus the discount = the interest on the discount.**

14. At what rate, simple interest, will the interest on £326 for $7\frac{1}{2}$ years be £110. 0*s.* 6*d.*?

15. In what time will £502. 13*s.* 4*d.* amount to £578. 1*s.* 4*d.* at $4\frac{1}{2}$ per cent. simple interest?

16. The time is 3 months, rate 4 p.c., simple interest £5. 10*s.* $10\frac{4}{5}d.$; what is the principal?

17. The rate is $4\frac{1}{2}$ p.c., simple interest £$5\frac{5}{8}$, principal £380. 4*s.* 2*d.*; find the time.

18. Find the amount of £6000 in 6 yrs. at 5 p.c. compound interest.

19. The present value of £826. 10*s.* due $3\frac{1}{2}$ years hence at simple interest is £725. Find the rate.

20. The discount on £$618\frac{1}{8}$ at 4 p.c. is £$80\frac{5}{8}$. Find the time at which it is due.

21. The difference between interest and discount at $2\frac{2}{5}$ p.c. for $2\frac{1}{2}$ years is £2. 11*s.* What is the sum due?

22. In what time will a given sum treble itself at 3 p.c. per annum, compound interest, payable half-yearly?

23. Three sums in A.P. are put out to compound interest for 2, 3, 4 years at 5 %. The amounts are then in A.P., but the sum which was greatest is now least. Find the ratio of the greatest to the least.

24. The interest on a certain sum of money for 9 months is £30. 18*s.*, and the discount at the same rate is £30. Find the sum of money and the rate.

25. Calculate in decimal form the terms of $(1 + \frac{3}{100})^5$. Find the compound interest on £100 for 5 yrs. at 3 per cent.

26. Money is put out to compound interest at $2\frac{1}{2}$ per cent.; show that it will have more than doubled itself in 29 years.

27. A sum of £1000 is lent on condition that it shall bear compound interest at 5 p.c. per ann. for the first 5 years, 10 p.c. per ann. for the next 5 years, and afterwards 15 p.c. per ann. What will the debt amount to in 20 years?

ANNUITIES.

328. An *Annuity* is a series of equal annual payments.

An Annuity may be *terminable*, *i.e.* it may cease at the end of a certain number of years; or it may be *perpetual*.

In calculating the accumulation of an annuity which has remained unpaid for a number of years, or the present value of an annuity, the question of interest comes in. In such questions simple interest is inapplicable, as it gives contradictory results. In the following investigations, therefore, it is assumed that *compound interest* is implied.

One form of annuity of £A is a property bringing in £A annually, such as a freehold estate bringing in rents to that annual amount. If the present value of such a property, *i.e.* the sum of money required to buy it, is £nA, the property is said to be worth *n years' purchase.*

329. *To find the present value of an annuity to continue for n years.*

Let the annuity be £A, r the interest on £1 for a year.

$\therefore$ amount of £1 in 1 year $= 1 + r = \mathsf{R}$.

£A would at end of 1 year amount to AR, at the end of 2 years to AR^2, and so on.

Let the present value be P.

By the end of n years P with its interest ought to amount to the same sum as the annual payments together with their interests.

At the end of the 1st a payment A is made.

By the end of the n years this will have been bearing interest for $n-1$ years.

$\therefore$ it will have amounted to AR^{n-1}.

Similarly the 2nd payment will have amounted to AR^{n-2}.

The last payment will be simply A.

Thus
$$\mathsf{PR}^n = \mathsf{AR}^{n-1} + \mathsf{AR}^{n-2} + \mathsf{AR}^{n-3} + \ldots + \mathsf{A}$$
$$= \frac{\mathsf{A}(\mathsf{R}^n - 1)}{\mathsf{R} - 1};$$
$$\therefore \quad \mathsf{P} = \frac{\mathsf{A}(1 - \mathsf{R}^{-n})}{\mathsf{R} - 1}.$$

Or thus:

The present value of the 1st payment $= \frac{\mathsf{A}}{\mathsf{R}}$,

..... 2nd ..... $= \frac{\mathsf{A}}{\mathsf{R}^2}$,

..... 3rd ..... $\frac{\mathsf{A}}{\mathsf{R}^3}$, and so on.

$\therefore$ present value of the annuity $= \frac{A}{R} + \frac{A}{R^2} + \frac{A}{R^3} + \ldots$ to n terms

$$= \frac{A}{R} \cdot \frac{1 - R^{-n}}{1 - R^{-1}} = \frac{A(1 - R^{-n})}{R - 1}.$$

The present value of a perpetual annuity

$$= \frac{A(1 - R^{-n})}{R - 1} \text{ where } n = \infty$$

$$= \frac{A}{R - 1} = \frac{A}{r}.$$

330. *To find the present value of an annuity £A to begin m years hence and to continue for p years.*

The 1st payment is to be made at the end of $m + 1$ years.

$\therefore$ present value of 1st payment $= \frac{A}{R^{m+1}}$;

.... 2nd ..... $= \frac{A}{R^{m+2}}$.

$\therefore$ present value of the annuity

$$= \frac{A}{R^{m+1}} + \frac{A}{R^{m+2}} + \ldots \text{ to } p \text{ terms}$$

$$= \frac{A}{R^{m+1}} \cdot \frac{1 - \frac{1}{R^p}}{1 - \frac{1}{R}} = \frac{A}{R^m} \cdot \frac{1 - R^{-p}}{R - 1}$$

$$= \frac{A}{r} \cdot (R^{-m} - R^{-m-p}).$$

The present value of an annuity to begin m years hence and to continue for ever is obtained by making p infinite:

$$\therefore \text{ present value} = \frac{AR^{-m}}{r}.$$

Examples. LI. b.

1. What is the present value of a perpetual annuity of £560 at 4 per cent.?

2. What is the present value of an annuity of £425 to continue for 48 years at 5 per cent.?

3. An estate is considered to be worth 25 years' purchase. What is the rate per cent.?

4. A perpetual annuity of £110 is bought for £2000; what is the rate of interest? How many years' purchase is it said to be worth?

5. Find the present value at $3\frac{1}{2}$ per cent. of an annuity of £100 to begin 8 years hence and continue for 15 years. Find also its present value if it is to be perpetual.

6. A person buys an estate yielding £300 a year for £6250. What rate per cent. is he obtaining for his money?

7. If I pay £2000 for the *reversion* of a property after 10 years [*i.e.* for a property which becomes mine after 10 years], what must be the rent from it in order that I may get 5 per cent. on my purchase-money?

8. How many years' purchase is a freehold estate worth (i) at $4\frac{1}{2}$ per cent., (ii) at 5 per cent.?

9. Find the present value of an annuity of £75 to begin 4 years hence and continue for 11 years, the rate being 5 per cent.

10. Find the present value of an annuity of £75 to begin now and last for 25 years at 5 per cent.

11. Find the cost of an annuity of £120 to run for 20 years at 5 per cent.

12. Find the present worth of a perpetual annuity of £441 to begin 2 years hence at 5 per cent.

13. I borrow a sum of money, and the debt is to be discharged by monthly payments of £10 for a year. If the rate of interest is 6 per cent. *per annum*, what do I borrow?

14. Find the present value of an annuity of £300 payable every year for 3 years at 5 per cent., the first payment being at the end of a year from now.

CHAPTER LII.

THE EXPONENTIAL THEOREM AND LOGARITHMIC SERIES.

331. By the Binomial Theorem, if n be numerically greater than 1,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\cdot\frac{1}{n} + \frac{nx(nx-1)}{\underline{2}}\cdot\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{\underline{3}}\cdot\frac{1}{n^3} + \dots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\underline{2}} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{3}} + \dots .$$

Also this result is true when $x = 1$.

$$\therefore \left(1+\frac{1}{n}\right)^{n} = 1 + 1 + \frac{1-\frac{1}{n}}{\underline{2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{3}} + \dots ;$$

$$\therefore \left\{1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\dots\right\}^x=\left\{\left(1+\frac{1}{n}\right)^n\right\}^x$$

$$=\left(1+\frac{1}{n}\right)^{nx}=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\dots.$$

This is true for all values of n greater than 1;

$\therefore$ by making n infinite we obtain the result

$$\left(1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\dots\right)^x=1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots.$$

The infinite series $1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\dots$ is denoted by e;

$$\therefore\ e^x=1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots.$$

This is the **Exponential Theorem.**

Cor. $a^x=e^{x\log_e a}$ as may be seen by taking logs. of both sides to the base e;

$$\therefore\ a^x=1+\frac{x\log_e a}{\lfloor 1}+\frac{x^2(\log_e a)^2}{\lfloor 2}+\dots.$$

Note. There are some points in connection with this proof which will probably not occur to the student and will certainly not be settled by him till he has read some of the subject of Convergency and Divergency.

332. *Logarithmic Series.*

$$a^x=1+x\log_e a+\frac{x^2(\log_e a)^2}{\lfloor 2}+\dots;$$

$$\therefore\ (1+y)^x=1+x\log_e(1+y)+\frac{x^2(\log_e 1+y)^2}{\lfloor 2}+\dots.$$

But by the Binomial Theorem (if y be numerically less than 1),

$$(1+y)^x=1+xy+\frac{x(x-1)}{1\,.\,2}y^2+\frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}y^3+\dots.$$

Comparing these two series for $(1+y)^x$ we get, by equating the coefficients of x,

$$\log_e(1+y)=y-\frac{1\,.\,y^2}{1\,.\,2}+\frac{1\,.\,2\,.\,y^3}{1\,.\,2\,.\,3}-\dots$$

$$=y-\frac{y^2}{2}+\frac{y^3}{3}-\dots$$

Similarly (or by writing $-y$ for y) $\log_e(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \ldots$,

and from these, by subtraction,

$$\log_e \frac{1+y}{1-y} = 2\left\{y + \frac{y^3}{3} + \frac{y^5}{5} + \ldots\right\}.$$

Also $$\log_e(n+1) - \log_e n = \log_e \frac{n+1}{n} = \log_e \frac{1 + \frac{1}{2n+1}}{1 - \frac{1}{2n+1}}$$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \ldots\right\}.$$

This series enables us to calculate $\log_e 2$ by putting $n=1$, $\log_e 3$ from $\log_e 2$, and so on.

Thus the logarithms of all numbers to base e can be calculated.

By Art. 243 we know that all these can be converted into *common* logarithms by multiplying by $\log_{10} e$, *i.e.* by $\frac{1}{\log_e 10}$, *i.e.* by ·43429....

This multiplier is known as the *modulus*.

333. *Calculation of* $\log_e 2$.

$$\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \ldots\right\};$$

$$\therefore \log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \ldots\right\}.$$

$\frac{1}{3} = \cdot3333333,$ $\frac{1}{3} = \cdot333333,$

$\frac{1}{3^3} = \cdot0370370,$ $\frac{1}{3}\cdot\frac{1}{3^3} = \cdot012346,$

$\frac{1}{3^5} = \cdot0041152,$ $\frac{1}{5}\cdot\frac{1}{3^5} = \cdot000823,$

$\frac{1}{3^7} = \cdot0004572,$ $\frac{1}{7}\cdot\frac{1}{3^7} = \cdot000065,$

$\frac{1}{3^9} = \cdot0000508,$ $\frac{1}{9}\cdot\frac{1}{3^9} = \cdot000006,$

$\frac{1}{3^{11}} = \cdot0000056;$

$$\therefore \tfrac{1}{2}\log_e 2 = \cdot346573.$$

Now $\frac{1}{11} \cdot \frac{1}{3^{11}}$ begins with 6 zeros;

$\therefore$ from this value of $\frac{1}{2}\log_e 2$ we get $\log_e 2$ correct certainly to 4 places.

$$\log_e 2 = \cdot 6931.$$

Example. Sum to infinity the series whose n^{th} term is $\frac{n^3}{\lfloor n}$.

$$n^{\text{th}} \text{ term} = \frac{n^3}{\lfloor n} = \frac{n^2}{\lfloor n-1} = \frac{1}{\lfloor n-1} + \frac{n^2-1}{\lfloor n-1}$$

$$= \frac{1}{\lfloor n-1} + \frac{n+1}{\lfloor n-2} = \frac{1}{\lfloor n-1} + \frac{3}{\lfloor n-2} + \frac{n-2}{\lfloor n-2}$$

$$= \frac{1}{\lfloor n-1} + \frac{3}{\lfloor n-2} + \frac{1}{\lfloor n-3}.$$

$$(n-1)^{\text{th}} \text{ term} = \frac{1}{\lfloor n-2} + \frac{3}{\lfloor n-2} + \frac{1}{\lfloor n-4},$$

.....

$$4\text{th term} = \frac{1}{\lfloor 3} + \frac{3}{\lfloor 2} + \frac{1}{\lfloor 1}.$$

$$3\text{rd term} = \frac{1}{\lfloor 2} + \frac{3}{\lfloor 1} + 1.$$

$$2\text{nd term} = \frac{1}{\lfloor 1} + 3.$$

$$1\text{st term} = 1.$$

$\therefore$ by adding up the columns and making n infinite, we get

$$\text{the series} = e + 3e + e = 5e.$$

Examples. LII.

1. Prove $\left\{1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots\right\}\left\{1 - \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \dots\right\} = 1.$

2. Find the value of $1 - \frac{2}{\lfloor 1} + \frac{2^2}{\lfloor 2} - \frac{2^3}{\lfloor 3} + \dots$ to infinity.

3. Find the value of $3^{-1} - \frac{1}{2} \cdot 3^{-2} + \frac{1}{3} \cdot 3^{-3} - \frac{1}{4} \cdot 3^{-4} + \dots$ to infinity.

4. Prove $\frac{x-y}{x} + \frac{(x-y)^2}{2x^2} + \frac{(x-y)^3}{3x^3} + \dots$ to infinity $= \log \frac{x}{y}$.

5. Prove $\frac{1}{e} = \frac{2}{\lfloor 3} + \frac{4}{\lfloor 5} + \frac{6}{\lfloor 7} + \dots$.

6. Expand $\log(1 + 3x + 2x^2)$ in ascending powers of x.

7. Expand $\log \frac{1-2x}{1-3x}$.

8. Find $\log_e 1{\cdot}04$ to 4 places by the series for $\log(1+x)$.

9. Expand $\log(1-x+x^2)$ in powers of x.

10. Sum the series $x^2+\frac{1}{x^2}-\frac{1}{2}\left(x^4+\frac{1}{x^4}\right)+\frac{1}{3}\left(x^6+\frac{1}{x^6}\right)\cdots$.

11. Given $\log_e 2 = {\cdot}693147$, find $\log_e 3$ from the series for $\log_e(n+1)-\log_e n$.

12. Given $\log_e 3 = 1{\cdot}098612$, calculate $\log_e 10$ by a series, and hence find $\log_{10} e$ to 5 places.

13. Sum the series $\frac{1}{\lfloor 2}+\frac{2}{\lfloor 3}+\frac{3}{\lfloor 4}+\ldots$ *ad inf.*

14. Sum the series $\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\ldots$ *ad inf.* if $x>1$.

15. Sum the series $1^2+\frac{2^2}{\lfloor 2}+\frac{3^2}{\lfloor 3}+\frac{4^2}{\lfloor 4}+\ldots$ *ad inf.*

16. Prove $\log_e\frac{1+x}{1-x}=2\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\right\}$ and find $\log_e\frac{2001}{1999}$ to 11 places.

17. Prove $1+\frac{1+2}{\lfloor 1}+\frac{1+2+3}{\lfloor 2}+\ldots$ *ad inf.* $=\frac{7e}{2}$.

18. Prove $\frac{1}{2}\log_e n=\frac{n-1}{n+1}+\frac{1}{3}\left(\frac{n-1}{n+1}\right)^3+\frac{1}{5}\left(\frac{n-1}{n+1}\right)^5+\ldots$.

19. Given $3^x = 7175{\cdot}37$, find x.

20. Prove $\log_e(n+1)-\log_e n=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\ldots\right\}$.

Hence show that $\log_e 13 = 2\log_e 2+\log_e 3+{\cdot}080042\ldots$.

21. Show that the difference of the logarithms of consecutive integers continually diminishes.

22. If $$x=y+\frac{y^2}{\lfloor 2}+\frac{y^3}{\lfloor 3}+\ldots \text{ to infinity,}$$

then $$y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots \text{ to infinity.}$$

23. If α, β are the roots of the equation $x^2+px+q=0$,

$$\log(1-px+qx^2)=(\alpha+\beta)x-\frac{\alpha^2+\beta^2}{2}x^2+\frac{\alpha^3+\beta^3}{3}x^3-\ldots.$$

24. Prove $\log_e(n+1)+\log_e(n-1)-2\log_e n$

$$=-2\left\{\frac{1}{2n^2-1}+\frac{1}{3(2n^2-1)^3}+\frac{1}{5(2n^2-1)^5}+\ldots\right\}.$$

25. Prove $n-\frac{1}{n}-\frac{1}{2}\left(n^2-\frac{1}{n^2}\right)+\frac{1}{3}\left(n^3-\frac{1}{n^3}\right)-\ldots=\log_e n$.

26. Prove $\dfrac{a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ldots}{b-1-\frac{1}{2}(b-1)^2+\frac{1}{3}(b-1)^3-\ldots}=\log_b a$.

27. If $f(p)$ denote the series

$$1+\frac{p}{\lfloor 1}+\frac{p^2}{\lfloor 2}+\frac{p^3}{\lfloor 3}+\dots,$$

show that $f(m+n)=f(m)\times f(n)$ identically.

Hence deduce the series for a^x in powers of x.

28. If x is numerically less than 1,

$$\frac{x}{1+x^2}+\frac{1}{3}\left(\frac{x}{1+x^2}\right)^2+\frac{1}{5}\left(\frac{x}{1+x^2}\right)^5+\dots=x-\frac{2x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}-\frac{2x^9}{9}+\dots.$$

29. Prove that $\log_e 3=1+\frac{1}{3\cdot 2^2}+\frac{1}{5\cdot 2^4}+\frac{1}{7\cdot 2^6}+\dots$ *ad inf.*

CHAPTER LIII.

INDETERMINATE COEFFICIENTS AND PARTIAL FRACTIONS.

ARBITRARY OR INDETERMINATE COEFFICIENTS.

334. If two algebraic expressions, which consist of terms containing integral powers of x, are identically equal (*i.e.* equal for all values of x) the coefficients of like powers of x in them are equal.

Suppose $a+bx+cx^2+dx^3+\dots=a_1+b_1x+c_1x^2+d_1x^3+\dots$ for all values of x.

These series are equal when $x=0$; $\therefore\ a=a_1$.

$\therefore\ bx+cx^2+dx^3+\dots=b_1x+c_1x^2+d_1x^3+\dots$.

This is true for all values of x.

$\therefore$ we may divide by x.

Thus $b+cx+dx^2+\dots=b_1+c_1x+d_1x^2+\dots$.

This is true when $x=0$; $\therefore\ b=b_1$.

Similarly for the other coefficients.

Example 1. *Sum the series* $1^2+2^2+3^2+\dots+n^2$.

Call the sum to n terms S.

Since $\mathrm{S}=0$ when $n=0$, there is no term independent of n.

$$\therefore\ \mathrm{S}=bn+cn^2+dn^3+en^4+\dots.$$

$$1^2+2^2+3^2+\dots+(n+1)^2=b(n+1)+c(n+1)^2+d(n+1)^3+e(n+1)^4+\dots.$$

But $1^2+2^2+3^2+\dots+n^2=bn+cn^2+dn^3+en^4+\dots$.

$\therefore$ by subtraction,

$$(n+1)^2=b+c(2n+1)+d(3n^2+3n+1)+e(4n^3+\dots)+\dots.$$

By equating coeffs. of like powers of n, we see that e and all subsequent coeffs. vanish.

$$b+c+d=1,\ 2c+3d=2,\ 3d=1;$$

$$\therefore\ c=\tfrac{1}{2},\ b=\tfrac{1}{6}.$$

$$\therefore\ S=\tfrac{1}{6}n+\tfrac{1}{2}n^2+\tfrac{1}{3}n^3=\frac{n}{6}[2n^2+3n+1]=\frac{n(n+1)(2n+1)}{6}.$$

Example 2. *Find the condition that x^3+qx+r may have a factor of the form $(x+a)^2$.*

Since one factor ends with a^2, the other must end with $\frac{r}{a^2}$.

$$\therefore\ x^3+qx+r=(x^2+2ax+a^2)\left(x+\frac{r}{a^2}\right)$$

$$=x^3+\left(2a+\frac{r}{a^2}\right)x^2+\left(\frac{2r}{a}+a^2\right)x+r.$$

$\therefore$ by equating coefficients, $0=2a+\frac{r}{a^2}$, $q=\frac{2r}{a}+a^2$;

$$\therefore\ 2a^3=-r;$$

$$\therefore\ q=-4a^2+a^2=-3a^2;$$

$$\therefore\ \left(\frac{q}{-3}\right)^3=\left(\frac{r}{-2}\right)^2;$$

$$\therefore\ 4q^3+27r^2=0.$$

335. *Expansion of fractions into series.*

The fraction $\frac{1}{1-x}$ may be expressed as an infinite series (x being less than 1), (1) by division, (2) by the Binomial Theorem, (3) by indeterminate coefficients.

(1) By division $\qquad \frac{1}{1-x}=1+x+x^2+x^3+\ldots.$

(2) By the Binomial Theorem

$$\frac{1}{1-x}=(1-x)^{-1}=1+x+x^2+x^3+\ldots.$$

(3) Let $\qquad \frac{1}{1-x}=a+bx+cx^2+dx^3+\ldots.$

Multiplying both sides by $1-x$, we get

$$1=a+bx+cx^2+dx^3+\ldots$$

$$-ax-bx^2-cx^2-\ldots$$

$$=a+(b-a)x+(c-b)x^2+(d-c)x^3+\ldots.$$

Equate coeffs. of like powers of x on the two sides.

Then
$$a=1;$$
$$b-a=0; \quad \therefore\ b=a=1.$$
$$c-b=0; \quad \therefore\ c=b=1.$$
$$d-c=0; \quad \therefore\ d=c=1\ldots.$$
$$\therefore\ \frac{1}{1-x}=1+x+x^2+x^3+\ldots.$$

By putting $-x$ for x, we obtain
$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots.$$

Similarly $$\frac{1}{1-px}=1+px+p^2x^2+p^3x^3+\ldots.$$

336. *A fraction of the form* $\dfrac{p+qx}{1+mx+nx^2}$ *may be expanded by indeterminate coefficients.*

Let $$\frac{p+qx}{1+mx+nx^2}=a+bx+cx^2+dx^3+\ldots.$$
$$\therefore\ p+qx=(1+mx+nx^2)(a+bx+cx^2+dx^3+\ldots)$$
$$=a+(b+ma)x+(c+mb+na)x^2+(d+mc+nb)x^3+\ldots.$$

By equating coefficients of like powers of x, we get
$$a=p, \qquad b+ma=q,$$
$$c+mb+na=0, \quad d+mc+nd=0, \text{ etc.}$$

From these equations we determine successively a, b, c, etc., and so obtain the required expansion.

Example 1. Expand $\dfrac{1+2x-3x^2}{1-2x+3x^2}$ in ascending powers of x.

Let $$\frac{1+2x-3x^2}{1-2x+3x^2}=a+bx+cx^2+dx^3+ex^4+\ldots.$$

Then $1+2x-3x^2=(1-2x+3x^2)(a+bx+cx^2+\ldots)$
$$=a+bx+cx^2+dx^3+ex^4+\ldots$$
$$-2ax-2bx^2-2cx^3-2dx^4-\ldots$$
$$+3ax^2+3bx^3+3cx^4+\ldots$$
$$=a+(b-2a)x+(c-2b+3a)x^2+(d-2c+3b)x^3+(e-2d+3c)x^4+\ldots.$$

Equate coefficients.
$$a=1.$$
$$b-2a=2; \quad \therefore\ b=2a+2=4.$$
$$c-2b+3a=-3; \quad \therefore\ c=2b-3a-3=8-3-3=2.$$
$$d-2c+3b=0; \quad \therefore\ d=2c-3b=4-12=-8.$$
$$e-2d+3c=0; \quad \therefore\ e=-16-6=-22, \text{ etc.}$$
$$\therefore\ \text{the fraction}=1+4x+2x^2-8x^3-22x^4-\ldots.$$

Example 2. The reverse process may be employed.

Find the fraction from which may be obtained the series

$$1+4x+2x^2-8x^3-22x^4-\dots,$$

assuming that some relation holds between every three consecutive coefficients.

Let $$\frac{a+bx+cx^2}{1+px+qx^2}=1+4x+2x^2-8x^3-22x^4\dots.$$

Then, by multiplication,

$$\begin{aligned}a+bx+cx^2=1&+4x+2x^2-8x^3-22x^4-\dots\\&+px+4px^2+2px^3-8px^4-\dots\\&+qx^2+4qx^3+2qx^4\dots.\end{aligned}$$

Equate coefficients.

Then
$$a=1, \quad \dots\dots(1)$$
$$b=4+p, \quad \dots\dots(2)$$
$$c=2+4p+q, \quad \dots\dots(3)$$
$$0=-8+2p+4q, \quad \dots\dots(4)$$
$$0=-22-8p+2q. \quad \dots\dots(5)$$

From (4) and (5), $2p+4q=8$, *i.e.* $p+2q=4$,

and $-8p+2q=22$.

From these simultaneous equations,

$$9p=-18;$$
$$\therefore\ p=-2;$$
$$\therefore\ q=3.$$

From (2), $b=4-2=2$.

From (3), $c=2-8+3=-3$.

From (1), $a=1$.

Substituting these values in $\frac{a+bx+cx^2}{1+px+qx^2}$ we obtain the required fraction

$$\frac{1+2x-3x^2}{1-2x+3x^2}.$$

Examples. LIII. a.

By the method of indeterminate coefficients sum the series:

1. $1^2+3^2+5^2+\dots$ to n terms.
2. $1\cdot2+2\cdot3+3\cdot4+\dots$ .....
3. $1\cdot2+3\cdot4+5\cdot6+\dots$ .....

4. Find the relation between p, q, r, that $x^4+2ax^3+px^2+qx+r$ may be a square.

5. Find the condition that x^3+px^2+qx+r may contain a factor of the form x^2-a^2.

6. Find a and b if x^3+ax^2+bx+1 is divisible by $(x+1)^2$.

7. Find a and b if $2x^3+3x^2+ax+b$ is divisible by $(x-1)(x-2)$.

8. Find c so that $3x^3+4x^2+cx+4$ may be divisible by x^2+1.

9. If $a+b+c=0$, $ax^5+bx^4+cx^3+cx^2+bx+a$ is divisible by x^2-1.

10. If $3x^2+8x+11$ can be put in the form
$a(x+1)(x+2)+b(x+2)(x+3)+c(x+3)(x+1)$, find a, b, c.

11. Express $5x-6$ in the form
$a(x-2)(x-3)+b(x-3)(x-1)+c(x-1)(x-2)$.

12. Express $x^4+5x^3+8x^2+2x+1$ in the form
$a+b(x+2)+c(x+2)^2+d(x+2)^3+e(x+2)^4$.

[On dividing by $x+2$, and then the quotient by $x+2$, and so on, we shall find that the successive remainders give the values of a, b, c ....]

Verify the result of the last question by putting $y-2$ for x in $x^4+5x^3+8x^2+2x+1$ and expanding in powers of y, *i.e.* in powers of $x+2$.

By indeterminate coefficients expand the following into series in ascending powers of x:

13. $\dfrac{1}{2+3x}$. **14.** $\dfrac{1-x}{1-2x-2x^2}$. **15.** $\dfrac{1-2x}{1-4x+x^2}$. **16.** $\dfrac{1-3x}{1-6x+8x^2}$.

Find the fractions which will expand into the following series:

17. $1-x^2-x^3+x^5+x^6-\ldots$. **18.** $1+2x+3x^2+4x^3+\ldots$

19. $2+5x+17x^2+65x^3+\ldots$. **20.** $1+3x+4x^2+7x^3+11x^4+\ldots$.

PARTIAL FRACTIONS.

337. The sum of two or more fractions can be expressed as a single fraction. It is often convenient to use the reverse process and resolve a single fraction into the sum of simpler fractions, **partial fractions** as they are called.

E.g. resolve $\dfrac{4-9x}{(1-2x)(1-3x)}$ *into partial fractions.*

Let $$\frac{4-9x}{(1-2x)(1-3x)}=\frac{\mathsf{A}}{1-2x}+\frac{\mathsf{B}}{1-3x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Here A and B are constants (*i.e.* quantities independent of x), whose values have to be determined.

From (1) $\quad 4-9x=\mathsf{A}(1-3x)+\mathsf{B}(1-2x)$.

Equating absolute terms we get $\mathsf{A}+\mathsf{B}=4$.

Equating coefficients of x we get $3\mathsf{A}+2\mathsf{B}=9$.

$$\therefore\ \mathsf{A}=1,\ \mathsf{B}=3;$$

$$\therefore\ \frac{4-9x}{(1-2x)(1-3x)}=\frac{1}{1-2x}+\frac{3}{1-3x}.$$

By division the right-hand side may be expressed as

$(1+2x+2^2x^2+2^3x^3+\ldots$ *ad inf.*$)+3(1+3x+3^2x^2+\ldots$ *ad inf.*$)$.

NOTE. When we had obtained $4-9x=\mathsf{A}(1-3x)+\mathsf{B}(1-2x)$, we might have proceeded by observing that this is true when $x=\frac{1}{2}$.

$$\therefore\ 4-\tfrac{9}{2}=\mathsf{A}(1-\tfrac{3}{2})+0\ ;\quad \therefore\ \mathsf{A}=1.$$

Also by putting $x=\frac{1}{3}$,

$$4-\tfrac{9}{3}=0+\mathsf{B}(1-\tfrac{2}{3})\ ;\quad \therefore\ \mathsf{B}=3.$$

338. *Resolve* $\dfrac{4+5x}{(1-x)^3}$ *into partial fractions.*

Let $$\frac{4+5x}{(1-x)^3}=\frac{\mathsf{A}}{1-x}+\frac{\mathsf{B}}{(1-x)^2}+\frac{\mathsf{C}}{(1-x)^3}.$$

Then $$4+5x=\mathsf{A}(1-x)^2+\mathsf{B}(1-x)+\mathsf{C}.$$

Put $x=1$, then $\mathsf{C}=9$.

Equate coefficients ; $\therefore\ \mathsf{A}=0$, and $\mathsf{A}+\mathsf{B}+\mathsf{C}=4$;

$$\therefore\ \mathsf{B}=-5\ ;$$

$$\therefore\ \frac{4+5x}{(1-x)^3}=\frac{9}{(1-x)^3}-\frac{5}{(1-x)^2}.$$

This question might be solved thus :

Put $1-x=y$, then $x=1-y$;

$$\therefore\ \frac{4+5x}{(1-x)^3}=\frac{4+5-5y}{y^3}=\frac{9}{y^3}-\frac{5}{y^2}=\frac{9}{(1-x)^3}-\frac{5}{(1-x)^2}.$$

339. We have to consider the case in which a factor of the denominator is a quadratic expression which does not contain simple factors ; *e.g.* $\dfrac{3x^2+7x+2}{(x+1)(x^2+2x+5)}$.

Let the fraction be equal to $\dfrac{\mathsf{A}}{x+1}+\dfrac{\mathsf{B}x+\mathsf{C}}{x^2+2x+5}$.

Then $$3x^2+7x+2=\mathsf{A}(x^2+2x+5)+(\mathsf{B}x+\mathsf{C})(x+1).$$

Equate coefficients. Then $\mathsf{A}+\mathsf{B}=3$, $2\mathsf{A}+\mathsf{B}+\mathsf{C}=7$, $5\mathsf{A}+\mathsf{C}=2$.

$$\therefore\ \mathsf{A}=-\tfrac{1}{2},\ \mathsf{B}=\tfrac{7}{2},\ \mathsf{C}=\tfrac{9}{2}\ ;$$

$$\therefore\ \text{the fraction}=\frac{7x+9}{2(x^2+2x+5)}-\frac{1}{2(x+1)}.$$

If we had taken $\dfrac{\mathsf{A}}{x+1}$ and $\dfrac{\mathsf{B}}{x^2+2x+5}$ as the partial fractions, we should have obtained 3 (inconsistent) equations to determine the 2 quantities A and B.

If the numerator is of lower degree than the denominator, the fraction is called a **proper fraction** ; otherwise it is an improper

fraction. If an improper fraction has to be resolved into partial fractions, it must first be reduced by division till the fractional part is a proper fraction.

For instance, $\dfrac{x^3+3x^2+x+4}{(x+1)(x+3)}$ must be reduced by division to the form $x-1+\dfrac{2x+7}{(x+1)(x+3)}$.

Examples. LIII. b.

Resolve into partial fractions

1. $\dfrac{2x}{x^2-1}$.
2. $\dfrac{x-4}{(x-2)(x-3)}$.
3. $\dfrac{4}{x^2-4}$.
4. $\dfrac{1}{x^2-4x-3}$.
5. $\dfrac{2x-1}{(1-x)(2-x)}$.
6. $\dfrac{2}{(x+1)(x+2)(x+3)}$.
7. $\dfrac{7x+5}{3x^2+4x+1}$.
8. $\dfrac{2x-3}{x(4x^2-1)}$.
9. $\dfrac{6x^2-4x-6}{(x-1)(x-2)(x-3)}$.
10. $\dfrac{4(x+10)}{x^4-16}$.
11. $\dfrac{1}{x^2(x^2-1)}$.
12. $\dfrac{16}{(x-2)(x+2)^2}$.
13. $\dfrac{x^2}{(x^2-1)(x-2)}$.
14. $\dfrac{x^3+x^2-x+3}{(x-1)(x+2)}$.
15. $\dfrac{x-1}{(x+2)(x+3)^2}$.
16. $\dfrac{25}{(x+2)^2(x^2+1)}$.
17. $\dfrac{x^3+4x^2+6x+1}{x(x^2+x+1)}$.

Resolve into partial fractions, expand in powers of x and give the general term of

18. $\dfrac{x}{1-5x+6x^2}$.
19. $\dfrac{10x+1}{1-x-12x^2}$.
20. $\dfrac{5}{x^2-3x-4}$.
21. $\dfrac{1}{(1-2x)(1-3x)(1-4x)}$.
22. $\dfrac{x}{(1-ax)(1-bx)(1-cx)}$. By expanding $\dfrac{1}{1-ax}$, etc., on both sides, multiplying together those on the left and equating coefficients, find the factors of

$a^2(b-c)+b^2(c-a)+c^2(a-b)$ and of $a^3(b-c)+b^3(c-a)+c^3(a-b)$.

23. $\dfrac{7(3x-1)}{(2-3x)(1+2x)}$.
24. $\dfrac{4x^3}{x^2-3x+2}$.
25. $\dfrac{2x+3}{x^2+x-2}$.
26. Resolve into partial fractions $\dfrac{3x-4}{(x-2)(x-1)^3}$.
27. ..... $\dfrac{x^2+1}{x^3-1}$.
28. The coefficient of x^n in the expansion of $\dfrac{x}{(1-x)(1-cx)}$ is $\dfrac{1-c^n}{1-c}$.

CHAPTER LIV.

MISCELLANEOUS EXAMPLES.

LIV. a.

1. By successive divisions by 5 obtain $\frac{1}{5}$, $\frac{1}{5^3}$, $\frac{1}{5^5}$, $\frac{1}{5^7}$ in decimals, and hence find the value of π correct to 4 decimal places from the formula

$$\frac{\pi}{4}=4\left(\frac{1}{5}-\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}-\frac{1}{7}\cdot\frac{1}{5^7}\right)-\frac{1}{239}.$$

2. If £·001 is taken to be a farthing, what percentage of a farthing is the error?

Read off £3·775 in pounds, shillings and pence.

3. Plot the points (6, 3), (10, 5). Join them by a straight line and from the diagram deduce the equation of this straight line. Also read off the abscissa of the point whose ordinate is 2·5, and the ordinate of the point whose abscissa is -7.

4. A certain old book is now worth £50, and is increasing in value at the rate of £5 a year. Assuming the value to continue rising at the same rate, find an expression for its value at any future time, and draw a diagram to show the value at any time. From your figure show the value of the book in 25 years.

5. Find a 4th proportional to 1·5, 3·9, ·8 by drawing a circle whose diameter is ABC, where AB = 3·9 and BC = ·8, and by drawing another circle whose centre is B and radius 1·5.

6. Draw the straight line graph $2x+\cdot7y=2\cdot5$ (inches). Find to one decimal place the length of the perpendicular to this line from the origin.

7. A manufacturer has priced certain lathes. The largest sells at £175, and the smallest at £40. He wishes to increase his prices so that the largest will sell at £200 and the smallest at £50. Assuming that the new price P and the old price Q are connected by the relation $Q=a+bP$, find the values of a and b, and, to the nearest pound, the new prices of lathes valued at £150, at £125. 10*s.* and at £78.

8. A rectangle has sides of length a and b. The error in measuring these is x per cent. and is very small. Prove that the error in the area is 0 or $\frac{abx}{50}$.

9. If 40 men can be arranged in a hollow square 2 deep, find the number in the front.

10. A man sets out to walk from A to B, a distance of 16 miles, at the rate of 3½ miles an hour. Three quarters of an hour afterwards a man sets out from B to meet him at the rate of 4 miles an hour. When and where do they meet?

LIV. b.

1. The sum of the first n even numbers $=\left(1+\frac{1}{n}\right)$ times the sum of the first n odd numbers.

2. How many submultiples has 225? How many has 144?

3. In the equation $3x^2+8xy+5y^2-12x-8y+1=0$, $x+h$, $y+k$ are substituted for x, y, and it is found that the coeffs. of x and y vanish in the new equation. Find h and k.

4. If $x+y+z=0$, $\Sigma(x^2)=-2\Sigma(xy)$ and $\Sigma\left(\frac{1}{x^2}\right)=\left\{\Sigma\left(\frac{1}{x}\right)\right\}^2$.

5. Find $\sqrt{10}$ by the ordinary process to 2 places of decimals. How many more places can be found by simple division? By means of the value of $\sqrt{10}$, find $\sqrt{\cdot 004}$ correct to 4 places.

6. Find the square of the series

$$1+\tfrac{1}{2}x+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot x^2+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}x^3+\ldots \text{ to infinity.}$$

7. Through the intersection of the straight lines whose equations are $2x-3y-7=0$ and $x-4y-1=0$ a straight line is drawn parallel to the axis of y. Find graphically its equation.

8. Express ·03125 as a power of 4.

9. An electric tram line with a single line of rails is $3\frac{1}{4}$ miles long. There are 13 trams available, and it is desired to arrange a 3-minute service of trams. How many crossing places must be provided, and what must be the speed of the trams?

10. Draw the graph of $y=x^2-3x+4$. Find the two points where it is cut by the graph of $y=mx$?

For what values of m are these coincident?

LIV. c.

1. Place the points (1, 2), (−3, 2), (2, −1), (−2, −3) on squared paper, taking the inch for unit. State the coordinates of the mid-points P, Q of the lines joining the first two points and the last two, and find the abscissa of the point where PQ cuts the axis of x.

2. If $r^2/r=g/289$, calculate v, having given that $r=4000$, $g=32\cdot2/5280$. Also show that $\frac{2\pi r}{v\times 60\times 60}=24$ approximately, where $\pi=3\cdot1416$.

3. Prove in any way that approximate values of $2^{\frac{1}{3}}$ and $2^{\frac{7}{12}}$ are $\frac{5}{4}$ and $\frac{3}{2}$ respectively. Find in each case whether the approximate or true value is the greater, and in which case the approximation is the closer.

4. Calculate $\log\frac{347\times\cdot231}{7\cdot62}$.

5. Find to one decimal place, on squared paper, the length of the tangent from the point (5, 6) to the circle $x^2+y^2-4x+4y=0$.

6. Give the remainder when $x^3-3x^2-18x+40$ is divided by $x-2$. Also when x^2+px+q is divided by $x-a$.

7. If $y=ax+b$, what numerical values must be given to a and b so that $y=-11$ when $x=-2$, and $y=1$ when $x=2$?

8. Find a point C in the line AB (a inches long), so that the sum of the squares on AC and CB shall be equal to a given square (side b inches). Find AC to the nearest hundredth of an inch when $a=4$, $b=3$.

9. If $a=x^k$, $b=x^l$, $c=x^m$, express as a power of x the fraction a^pb^q/c^r.

10. Having given that $10^x=2$, calculate 10^{2x}, 10^{4x}, 10^{8x}, 10^{16x}; and hence show that x lies between $\frac{4}{16}$ and $\frac{5}{16}$.

Show without using tables that $\log_{10}2$ lies between ·25 and ·32.

LIV. d.

1. Find the least fraction which is greater than $\frac{12}{17}$ and has its denominator greater by 13 than its numerator.

2. Find the sum of all numbers between 400 and 1000 which are divisible by 17.

3. Show that if all the terms of a G.P. are positive, the sum to infinity cannot be less than 4 times the 2nd term.

4. Find which term in the expansion of $(3x+5)^{20}$ has the greatest coefficient.

5. Find the value of $9^{\frac{1}{6}} \times 3^{\frac{1}{12}} \times 2^{\frac{1}{24}}$.

6. The number of births in a town is 25 in every thousand of the population annually, and the deaths 20 in a thousand. In how many years will the population double itself?

7. If $f(n) = \frac{1}{6}n(n+1)(2n+1)$, find $f(n+1) - f(n)$.

8. Find a positive value of x satisfying the equation $2^x = 7 \times 2^{\frac{1}{x}}$ approximately, so that the error does not exceed one per cent.

9. Solve the equation $2x^2 + y^2 - 2xy - 2y + 1 = 0$ as an equation for y, and ascertain for what values of x, y is real.

10. If $$(1+x)^n = 1 + n_1 x + n_2 x^2 + n_3 x^3 + \dots,$$
find the value, when $n = 9$, of $1 - n_2 + n_4 - n_6 + n_8$.

LIV. e.

1. Evaluate $2^{-2}3^{\frac{1}{2}}x^{\frac{5}{6}} + 2^{-3}3^{-\frac{1}{2}}x^{\frac{2}{3}} - 10(27x)^{-\frac{1}{6}}$ when $x = 64$.

2. Find the square root of $2ab - ax - bx - 2\sqrt{ab(ab - ax - bx + x^2)}$.

3. Prove that the cube of a number of n digits cannot have less than $3n-2$ nor more than $3n$ digits.

4. A walks m miles in n hours; B walks $7n$ miles in $\frac{1}{2}m$ hours; the difference of their rates of walking is $\frac{1}{2}$ mile per hour. Find the rate at which each walks.

5. Show that when $4a + 4b + 1 = 0$, the equations
$$x^2 - 2ax + b = 0, \quad x^2 - 2bx + a = 0,$$
have a common root.

6. If x, y, z are in H.P., prove that $\frac{x}{y+z}$, $\frac{y}{x+z}$, $\frac{z}{x+y}$ are in H.P.

7. A number is expressed by 2 digits, whose sum is 8, with a decimal point between them, and its double is one less than the number obtained by interchanging the digits: what is the number?

8. In the expansion of $(a+b)^{10}$ if $a+b=1$, show by logarithms that the first term > the sum of all the remainder if a is greater than (about) ·933.

9. The interest on a sum of P pounds for a certain time is I pounds and the discount at the same rate of (simple) interest is D pounds. Show that
$$\frac{1}{P} = \frac{1}{D} - \frac{1}{I}.$$

10. Draw the graph of $x^2 - y^2 - 4x - 2y + 4 = 0$, and find where it is met by the line $x=1$, and by the line $y=2$.

LIV. f.

1. A and B are two stations on a line 90 miles apart. At the same instant one train passes through A towards B and another through B towards A, with different but constant speeds. They pass each other at C, and AC is 10 miles longer than BC; also, the first reaches B half-an-hour before the second reaches A. Find their speeds by graphical methods.

2. Explain how to test multiplication of numbers by 'casting out the nines.' Two numbers A, B give remainders 5 and 7 respectively when divided by 9: find the remainder when their product is divided by 9.

3. Find the number of digits in the 250th term of $2+4+8+16+\ldots$.

4. Find log 83 to the base 19 as accurately as your tables permit.

5. Show that the straight line joining the points (24, 16) and $(-21, -14)$ passes through the origin, and find the coordinates of the points of trisection of this line.

6. A man walks a third as far again as a boy in a given time. His stride is 5 centimetres longer than the boy's, and he takes 125 strides per minute, while the boy takes only 100. At what rate in kilometres per hour does each walk?

7. In how many ways could a party of 5 scouts be selected from 20 men? In how many ways could the 20 men be formed into 4 parties of 5 scouts, to proceed in different directions?

8. Sum to n terms the series whose n^{th} term is $an+2^{n-1}$.

9. What values must a, b, c have that $3x^2+10x+3$ may be equal to $a(x-b)(x-c)$ for all values of x?

10. If $a+b+c=p$, then $a(b+c)+b(c+a)+c(a+b)$ cannot exceed p^2.

LIV. g.

1. Find the value of $\dfrac{x+p}{q}-\dfrac{x+q}{p}$, when $x=\dfrac{q^2}{p-q}$.

2. Calculate to 4 significant figures:

(i) $\dfrac{52{\cdot}45\times 378{\cdot}4\times {\cdot}02086}{87{\cdot}32\times {\cdot}5844}$. (ii) $(1{\cdot}246)^{4{\cdot}125}$.

3. Simplify $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}$, and find its value to 3 places.

4. A and B start together in a long distance race. For a quarter of an hour A goes at the rate of x yds. per second, and B $2x$ miles per hour: and then A is leading by 100 yds. Find x.

5. If $\dfrac{a}{b}=\dfrac{c}{d}$, and x be a 3rd proportional to a and b, y to b and c, prove that the geometric mean of x and y = that of c and d.

6. A meteor is seen at a height of 450 feet above the ground, one second later it is at a height of 328 feet, and 2 seconds after the 1st observation it is at a height of 174 feet. Determine the values of a, b, c, if the formula at^2+bt+c represents for all values of t the height of the meteor t seconds after the 1st observation.

7. $f(x)=\mathrm{A}\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\mathrm{B}\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\mathrm{C}\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$.

Find $f(a)$, $f(b)$, $f(c)$.

8. Find by the Binomial Theorem the first 20 figures of $\frac{1}{99^6}$.

9. Expand $\log(2+x-x^2)$ in ascending powers of x, if $x<1$.

10. Prove by Mathematical Induction that $(3n+1)7^n-1$ is always divisible by 9.

LIV. h.

1. Prove that $\frac{a+p}{b+p}$ lies between $\frac{a}{b}$ and $\frac{a+2p}{b+2p}$, where a, b, p, q are all positive quantities.

2. What is the necessary condition that the sum to infinity of the series $\frac{a^2}{b}+\frac{a}{c}+\frac{b}{c^2}\ldots$ may be finite? Find the sum when this condition is satisfied.

3. A certain football club consists of 20 members, of whom 11 can only play forward, and 9 can only play back. In how many ways can a team of 8 forwards and 7 backs be selected?

4. Find the coeff. of x^{17} in $\left(\frac{x^2}{2}-\frac{2}{x}\right)^{16}$.

5. If α, β are the roots of the equation $ax^2+bx+c=0$, construct the quadratic whose roots are $\frac{1-\alpha}{1+\alpha}$, $\frac{1-\beta}{1+\beta}$.

6. Find approximately by logarithms the value of the incommensurable root of $3^{2x}-5\times 3^x+6=0$.

7. Write down the first negative term in the expansion of $(1+x)^{\frac{2\cdot7}{6}}$, x being positive.

8. If $x<a$, and $x<b$, resolve $\frac{1}{ab-(a+b)x+x^2}$ into partial fractions and expand each of them. Prove that the coeff. of x^{n-1} is $\frac{a^{-n}-b^{-n}}{b-a}$.

9. Prove that 14641 is a square number in any scale.

10. The consumption of coal by a locomotive $\propto$ the square of the velocity. When the speed is 16 miles an hour, the hourly consumption is 2 tons. If the price of coal be 10*s.* a ton, and the other expenses be 11*s.* 3*d.* an hour, find the most economical speed, and the least cost for a journey of 100 miles.

LIV. k.

1. Find correct to 2 places $\sqrt[7]{2\cdot718}$ and $\sqrt[5]{673\cdot8}$.

2. If x^3+2ax^2+bx+c is divisible by x^2+ax+d, find b and c in terms of a and d.

3. Find the sum of all the numbers between 600 and 2000 which are divisible by 7.

4. How many numbers are there consisting of five different digits arranged in ascending order of magnitude?

5. If $a+b+c=0$, then $2(a^5+b^5+c^5)=5abc(a^2+b^2+c^2)$.

6. Draw the graphs of $y^2=12x$ and $y=x+3$, and find where they meet.

7. Find two successive terms in the expansion of $(1+x)^{39}$ such that their coefficients are in the ratio 5 : 1.

8. Given the height, the volume of a cylinder varies as the square of the radius. Find the volume of a cylinder whose radius is 5 inches, if the volume of a cylinder of the same height and radius 3 inches is 282·744 cubic inches.

9. Show that the sum of the series

$$\frac{5}{6}+\frac{5\,.\,7}{6\,.\,9}+\frac{5\,.\,7\,.\,9}{6\,.\,9\,.\,12}+\ldots \text{ to infinity}$$

is $3\sqrt{3}-2$. (A binomial expansion with index $-\frac{3}{2}$.)

10. Sum the series

$$\frac{1}{5}-\frac{1}{2}\cdot\frac{1}{5^2}+\frac{1}{3}\cdot\frac{1}{5^3}-\frac{1}{4}\cdot\frac{1}{5^4}+\ldots \text{ to infinity.}$$

LIV. l.

1. Prove that if a number of 2 digits = 4 times the sum of its digits, the reversed number = 7 times the sum of the digits.

2. If $x+y : x-y = a : b$, express the ratio $x^2+y^2 : x^2-y^2$ in terms of a and b.

3. Find the coefficient of x^{10} in $\{(1+x)(1+x^2)\}^8$.

4. Find the greatest value of $\dfrac{x+2}{4x^2+16x+25}$.

5. If $ma^2+nc^2 : pa^2+qc^3 = mb^2+nd^2 : pb^2+qd^2$, then either $a : b = c : d$, or $m : n = p : q$.

6. Find 4 numbers in A.P. whose sum is 32, the harmonic mean between the 2nd and 3rd being $7\frac{1}{2}$.

7. The 1st and 2nd of the 3 digits by which a perfect square is expressed are 1 and $2n$ respectively; find the 3rd digit, and show that, in any scale, a perfect square is obtained by reversing the digits.

8. Solve $\dfrac{x}{a}-\dfrac{y}{b}=2$, $\sqrt{2a(b-y)}=\sqrt{b}(\sqrt{a}+\sqrt{x})$.

9. Show that

$$(1-x^2)^n=(1+x)^{2n}-2nx(1+x)^{2n-1}+\frac{2n(2n-2)}{1\,.\,2}x^2(1+x)^{2n-2}-\ldots.$$

10. Find the coefficient of x^{12} in the product of

$$\frac{1+x^3}{(1-x^2)(1-x)} \text{ and } 1-x+x^2.$$

LIV. m.

1. A tricycle, going 5 miles an hour, passes a milestone, and 14 minutes afterwards a bicycle, going in the same direction at 12 miles an hour, passes the same milestone; find graphically when and where the bicycle will overtake the tricycle.

2. Solve the equation $x(x^2-4)=a(a^2-4)$.

3. If the sum of the first p terms of an A.P. $=0$, the sum of the next q terms $=\dfrac{-a(p+q)q}{p-1}$.

4. If a, b, c are in A.P.; a, mb, c in G.P.; prove that a, m^2b, c are in H.P.

5. Determine the scale of notation with highest radix which requires 4 digits to express the number five hundred; and find what the 4 digits will be.

6. Find the coefficient of x^8 in the expansion of $\dfrac{1}{1+x+x^2+x^3}$.

7. If $\sqrt{\mathsf{N}}=a+x$, where x is very small, prove that approximately
$$\sqrt{\mathsf{N}}=a\,.\,\frac{3\mathsf{N}+a^2}{\mathsf{N}+3a^2}.$$

8. Sum the series $1\,.\,3+2\,.\,4+3\,.\,5+\ldots+n(n+2)$.

9. Prove $\log_e a-\log_e x=\dfrac{a-x}{a}+\dfrac{1}{2}\left(\dfrac{a-x}{a}\right)^2+\dfrac{1}{3}\left(\dfrac{a-x}{a}\right)^3+\ldots$.

10. Resolve into partial fractions $\dfrac{x^2-x+1}{(x^2+1)(x-1)^2}$.

LIV. n

1. Insert 19 Arithmetic and also 19 Harmonic means between 2 and 3, and show that if A be any arithmetic mean and H the corresponding harmonic mean, $\mathsf{A}+\dfrac{6}{\mathsf{H}}=5$.

2. The number of permutations of *abbccc* in which the *b*'s come together $=\frac{1}{3}$ of the whole number of permutations.

3. If a, b, c, d are in G.P., $\log_a \mathsf{N}$, $\log_b \mathsf{N}$, $\log_c \mathsf{N}$, $\log_d \mathsf{N}$ are in H.P.

4. Resolve into partial fractions $\dfrac{2b^2}{(x-a+b)(x-a-b)(x-a)}$.

5. Find the relation between a, b, and c in order that x^3+3ax^2+bx+c may be a cube.

6. Expand $(1-x)^{-2}$, $(1-x)^{-3}$, $(1-x)^{-4}$ giving the coefficient of x^n in each case.

7. Find sq. rt. of $23t314$ in the scale of 12.

8. Draw a circle 2·6 inches in radius and take a point O 4·4 inches from its centre. Consider a straight line OPQ passing through O and cutting the circle in P, Q. Draw this line in 4 positions including the one in which P and Q coincide, and by measuring OP and OQ in each position, prove that the rect. OP, OQ is constant. Find an equation connecting r the radius of the earth, h the height of a mountain, and d the distance of the sea-horizon as seen from the mountain-top.

9. An approximation to $\dfrac{1}{365}$ is $\cdot 002\left(1+\dfrac{1}{3}+\dfrac{1}{30}+\dfrac{1}{300}\right)$. To how many places is it correct?

10. The height of a mountain in feet is given by
$$\mathsf{H}=49000\left(\frac{\mathsf{R}-r}{\mathsf{R}+r}\right)\left(1+\frac{\mathsf{T}+t}{900}\right),$$
where R, r are the observed heights of the barometer in inches at the bottom and top of the mountain, and T, t are the observed temperatures

at the bottom and top. The following observations were made at the bottom and top of a certain mountain:

	Barometer.	Thermometer.
At bottom	29·60 in.	67°
At top	25·35 in.	30°

Find the height of the mountain and roughly check your result by any method which occurs to you.

LIV. p.

1. If x is real, show that $\dfrac{x^2-4x-20}{x-7}$ cannot have a real value between 8 and 12.

2. The ratio of B's age to A's age is now 7 : 4, but in 10 years' time it will be 19 : 13. Find their ages.

3. Draw the graph of the equation $2x^2+5xy-3y^2-5x+6y-3=0$, after resolving into factors the expression in x, y.

4. During any year the excess of births over deaths causes an increase of h per cent. of the population, and at the end of the year a fixed number A of people emigrate. Prove that at the end of n years a population P becomes $b^n\mathrm{P}-\dfrac{b^n-1}{b-1}\mathrm{A}$, where $b=1+\dfrac{h}{100}$.

5. Find, without assuming any formula, the number of different ways in which n men may stand in a row.

If two specified men are neither of them to be at either end of the row, show that the number of arrangements is $(n-2)(n-3)\lfloor n-2$.

6. Eliminate a from the equations $x=\log_a b$, $y=\log_a c$.

7. Solve the equation $24^3=16^{x-1}$.

8. Prove by the Binomial Theorem that money at 5 per cent. compound interest more than doubles itself in 15 years.

9. The middle term of $(1+x)^{2n}$ is $\dfrac{1\,.\,3\,.\,5\,\ldots\,(2n-1)}{\lfloor n}(2x)^n$.

10. Prove $\dfrac{1}{1\,.\,2\,.\,3}+\dfrac{1}{3\,.\,4\,.\,5}+\dfrac{1}{5\,.\,6\,.\,7}+\ldots=\log_e 2-\frac{1}{2}$.

LIV. q.

1. Prove by Mathematical Induction that the sum

$1+3+9+27+\ldots$ to n terms is $\frac{1}{2}(3^n-1)$.

2. Two casks A and B contain mixtures of wine and water, A in the ratio of 7 to 4, and B in the ratio of 9 to 2. In what ratio must liquid be drawn from each cask to give a mixture in the ratio of 3 to 1?

3. A quantity of water contained in a cubical cistern is found to lose by evaporation ·04 of its volume in a day. The depth is 6 feet, and a cub. ft. of water weighs 1000 oz. Assuming the loss to be only through evaporation, what weight of water will be left at the end of 10 days?

4. If the expression $x(1-5x+6x^2)^{-1}$ be expanded in powers of x, prove that the coeff. of x^r is 3^r-2^r. [Resolve into partial fractions.]

5. Expand $(1+x^2)^{-\frac{1}{2}}$ when $x>1$.

6. If $x+\frac{1}{x}=y$, express $x^3+\frac{1}{x^3}$ and $x^5+\frac{1}{x^5}$ in terms of y.

7. The duration of a railway journey varies directly as the distance and inversely as the velocity. The velocity varies directly as the sq. root of the quantity of coal used per mile and inversely as the number of carriages. In a journey of 25 miles in half an hour with 18 carriages, 10 cwt. of coal is required. What will be consumed in a journey of 15 miles in 20 minutes with 20 carriages?

8. Write down the n^{th} term of the series 8, 15, 24, 35, ... and find the sum of n terms. $[8=3+5,\ 15=3+5+7.]$

9. Expand $\log(x^2+4x+3)$ in ascending powers of x, supposing x to be less than 1.

10. If

$$(1-x)^{-1}(1-x^2)^{-p}(1-x^3)^{-q}(1-x^4)^{-r}=1+x+2(px^2+qx^3+rx^4)+\dots,$$

show that $p=\frac{1}{2}\ q=\frac{1}{3}\ r=1.$

LIV. r.

1. How many square yards are there in a courtyard x ft. y inches long and y ft. x inches wide?

If a part of it x ft. x inches by y ft. y inches is covered with grass, how many tiles $\frac{x-y}{2}$ inches square will cover the remainder?

2. Show that one root is the square of the other in the equation

$$x^2-x=2+\sqrt{5}.$$

3. If $x=\log_a bc$, $y=\log_b ca$, $z=\log_c ab$, prove that $xyz=x+y+z+2$.

4. Find the value, when $x=\frac{1}{4}\sqrt{3}$, of

$$\frac{1+2x}{1+\sqrt{1+2x}}+\frac{1-2x}{1-\sqrt{1-2x}}.$$

5. B and C together can finish a piece of work in $\frac{1}{3}$ of the time that A would take; C and A together in $\frac{1}{4}$ of the time that B would take. Compare the times in which A and B can separately do the work.

6. Find graphically the value of $\sqrt{22+12\sqrt{2}}$.

7. The hypotenuse of a right-angled triangle is less than the sum of the other two sides by 8 ft., and the area $=180$ sq. ft. Find all the sides.

8. There are 8 stations on a railway. How many different sorts of *single* tickets of each class must be printed so as to include all possible journeys on that line?

9. Draw the graph of $x^2-6x+41$ and find its minimum value.

10. Prove that the arithmetic mean of any two positive quantities is greater than the harmonic mean.

LIV. s.

1. Simplify $(27a^{\frac{3}{4}})^{\frac{1}{3}}\times(9a^{-\frac{3}{2}})^{-\frac{1}{2}}$.

2. The true discounts on a sum of money for 3 months at 4 and 5 per cent. differ by 1*s.* 8*d.* Find the sum of money.

3. Transform 235757 from scale 8 to scale 12.

4. If $(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2$, prove that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

5. Given that the time in seconds of a beat of a pendulum is $\pi\sqrt{\frac{l}{g}}$, where l is the length in feet, find the length of a pendulum which beats seconds if $g=32{\cdot}2$ and $\pi=3{\cdot}142$.

6. Find the length of the common chord of two equal circles of radius 5 inches if the centres are 9·8 inches apart.

7. Simplify $(3\sqrt{3}-5)^{\frac{1}{2}}\div(2-\sqrt{3})$, and find its value to 3 places.

8. In how many ways can 10 boys be arranged so that a particular pair never sit next to each other (1) when they form a row, (2) when they form a ring?

9. A starts $37\frac{1}{2}$ minutes before B to walk $11\frac{3}{4}$ miles, the first $5\frac{1}{4}$ of which are uphill and the rest downhill. Downhill their speeds are respectively $\frac{1}{4}$ and $\frac{1}{5}$ mile per hour faster than uphill, and they finish their journey at the same moment, though A reaches the hilltop $22\frac{1}{2}$ minutes before B. What are their initial speeds?

10. If a, x, y, b are in A.P., and c^3, x, y, d^3 in G.P., show that

$$a+b=cd(c+d).$$

LIV. t.

1. Draw on squared paper the line $2x+0{\cdot}7y=2{\cdot}5$ (unit = 1 inch). Find to 1 decimal place the length of the perpendicular to it from the origin.

2. Find r, if ${}^nP_r=120\,{}^nC_{n-r}$.

3. In a certain year a rise of 5·37 per cent. in the female population, and of 5·61 in the male, produced a rise of 5·5 per cent. on the whole. Compare the number of males and females at the beginning of the year, and also at the end.

4. Write down the continued product of the n factors

$$1+x_1,\ 1+x_2,\ 1+x_3,\ \ldots\ 1+x_n,$$

and deduce the expansion of $(1+x)^n$.

5. Find the 12th term and middle term of $\left(a-\frac{1}{a}\right)^{16}$.

6. A pendulum 3 ft. long is observed to make 700 beats in 671 seconds. If the time of a beat is $\pi\sqrt{\frac{l}{g}}$ seconds, where l = length of pendulum in feet, find the value of g.

7. Find $\sqrt[3]{128}$ to 5 places of decimals.

8. In how many years will £20 become £36 at 5 per cent. compound interest?

9. The tension of a string with one end fixed and the other attached to a moving body varies directly as the square of the body's velocity and inversely as the length of the string. If the string will just bear the strain of a body moving with velocity 10 feet a second when the length of string is 3 feet, find the greatest velocity of the same body which the same string shortened to one foot will stand.

10. If $\phi(x)=x+\frac{1}{x}$, find $\phi(y+\sqrt{y^2-1})$.

LIV. u.

1. Show that approximately $1000001^2 : 1000000^2 = 1000002 : 1000000$.

2. Prove that $(2+\sqrt{3})^{\frac{3}{2}} + (2-\sqrt{3})^{\frac{3}{2}} = 3\sqrt{6}$.

3. Solve the equation $3^x - 3^{x-1} = 486$.

4. The birth-rate in a town is 20 in a thousand, the death-rate is 14 in a thousand annually; in how many years will the population double itself?

5. In how many different ways can 8 persons be formed into 2 groups?

6. By resolving into partial fractions expand the fraction $\frac{1}{x^2-5x+6}$ in a series:

(i) of the form $p_0 + p_1 x + p_2 x^2 + \dots$,

(ii) of the form $q_0 + \frac{q_1}{x} + \frac{q_2}{x^2} + \dots$,

and show that $q_{n+1} = 6^n p_{n-1}$.

7. Transform 106344, which is in the scale of 7, to the scale of 10; and find its square root in each scale. Check your results by converting one of the square roots into the other scale.

8. Sum to n terms the series whose r^{th} term is $2r - 2^r$.

9. Find the number of digits in $2^{19} \times 3^{17}$, and the first 4 digits of the product.

10. Find the coefficient of x^{4r} in the expansion of $\log(1 + x + x^2 + x^3)$.

LIV. v.

1. The incomes of A and B are as 6 to 5, their expenditure as 3 to 2. Each saves £270 a year: find their incomes.

2. Find a value of x which will make $x^4 + 2ax^3 + b^2x^2 + c^3x + d^4$ a square.

3. Find the scale of notation in which the common number 1156 is represented by 961.

4. Which term of the series 1, 3, 5 ... is a mean proportional between the 23rd and 63rd? Find also how many terms of the series have their sum equal to one-fourth of the sum of the first 20 terms.

5. A man arranges to pay off a debt of £3600 by 40 annual instalments, which form an A.P. When 30 of the instalments are paid he dies, leaving a third of the debt unpaid. Find the value of the first instalment.

6. Expand $(\frac{5}{3} + x)^{\frac{7}{2}}$, and find which term has the greatest numerical coefficient.

7. Sum to infinity the series $x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \dots$.

8. Find the coefficient of x^6 in $(1 + x + x^2)^{-3}$.

9. A man's net income is 10 per cent. less than his gross income, which is derived partly from land and partly from personal property. He pays income tax at $8d.$ in the £ on the whole, and also local rates on the income from land. If his whole income had been derived from land, his net income would have been one ninth less. What are the local rates per £, and in what proportion is his gross income derived from the two kinds of property?

10. Draw the graph of $\frac{x^2 - 13}{x^2 + 5x + 6}$.

LIV. w.

1. Simplify $\left(\frac{x^{p+q}}{x^q}\right)^p \div \left(\frac{x^q}{x^{q-p}}\right)^{p-q}$ and $2^{\frac{1}{2}} . 16^{\frac{1}{8}} + 2^{-\frac{1}{2}} . 4^{\frac{5}{4}} + 9^{\frac{7}{8}} . 3^{\frac{1}{4}}$.

2. A rod whose length is 5 feet is broken into 3 pieces, out of which a right-angled triangle can be formed whose area is 120 sq. inches. Find the length of its sides.

3. If $xy+3 : xz+1 = y^2+3 : yz+1$, show that $y=x$ or $y=3z$.

4. Find the number of digits in 2^{64}, and if $a^1 . a^2 . a^3 \ldots a^n = p$, find an equation giving n in terms of p and a.

5. Solve the equations $2^x = 8^{y+1}$, $9^y = 3^{x-9}$.

6. Decompose into partial fractions $\frac{6x^2-x-1}{(x^2+1)(x-2)}$.

7. A man spends on charitable objects an annual amount proportional to the square of his income, and spends £35 more when his income is £1200 than when it is £900 per annum. Find his charitable expenditure in each case.

8. Write down the 2 middle terms in the expansion of $\left(x-\frac{1}{x}\right)^{15}$, and find how many terms of $(1-3x)^{\frac{7}{3}}$ have positive signs.

9. Prove that the sum of two numbers is divisible by 11, if every digit which stands in an even place in one stands in an odd place in the other.

10. Sum the infinite series whose n^{th} term is $\frac{n}{\lfloor n-1}$.

Prove that it is twice the square of the series

$$1+\frac{1}{2\lfloor 1}+\frac{1}{2^2\lfloor 2}+\frac{1}{2^3\lfloor 3}+\ldots \text{ to infinity.}$$

LIV. x.

1. Simplify $\frac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}} - (a^{\frac{1}{6}}-b^{\frac{1}{6}})^4$.

2. Express $\sqrt{158+60\sqrt{6}}$ as the sum of 2 surds, and find its value correct to 3 decimal places.

3. Sum the series $(5a+7b)+(8a+5b)+(11a+3b)+\ldots$ to 18 terms.

4. Show that if a and b are such that the sum of the squares of the 3 arithmetic means inserted between a and $b=a^2+b^2$, then the sum of the cubes of these means $=\frac{1}{2}(a+b)^3$.

5. $\log 2^{x+3} = 1{\cdot}2222$. Find x.

6. Find by logarithms the value of $\sqrt{13\sqrt{5}\div\sqrt{7}}$.

7. Find the number of ways in which 5 boys and 3 girls may be arranged in a row (1) so that the girls may be all together, (2) so that no two girls may be together.

8. If $(1-x)^{-m} = c_0+c_1x+c_2x^2+\ldots$, prove that

$$c_0+c_1+c_2+\ldots+c_k = \frac{\lfloor m+k}{\lfloor m \lfloor k},$$

x being less than 1, and m, k being positive integers.

9. Find the least possible value of $4x+\frac{9}{x}$.

10. If the expression $x(1-5x+6x^2)^{-1}$ be expanded in powers of x, prove that the coefficient of x^r is 3^r-2^r.

LIV. y.

1. Find the ratio of the angle of a regular polygon of $2n$ sides to the angle of a regular polygon of n sides. Check your result by taking $n=4$. When is the ratio equal to 2?

2. A bicyclist started 3 minutes late to keep an appointment at a place 15 miles off. By going at a rate half a mile an hour faster than he need otherwise have done, he arrived at the proper time. At what rate did he ride?

3. In how many ways can 12 persons be divided into 3 sets of 4 each, one set to play lawn tennis, one to play croquet, and one to play bowls?

4. Find the coefficient of x^3 in $(32x-\frac{1}{2})^{20}$.

5. Convert 222·22 from the scale of 5 to the scale of 10.

6. $3^x=7175$. Find x.

7. Two casks, each containing 20 gallons, are filled, one with water, the other with spirit; x gallons are drawn from each cask, mixed, and the casks are again filled up with the mixture. When this is done a second time, it is found that the quantity of spirit in one cask is to the quantity of spirit in the other as 5 to 3. Find x.

8. If x be real, between what limits does $\frac{2x^2+6x+3}{2x+1}$ lie?

9. Find the coefficient of x^r in the expansion of $\frac{a-bx}{e^x}$.

10. How many years must elapse before a sum of money doubles itself at 3 per cent. compound interest?

LIV. z.

1. If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$, prove $\frac{x}{a}=\frac{y}{b}$.

2. If a number consists of n figures, its square cannot contain more than $2n$ or less than $2n-1$ figures.

3. Give, from a table of common logarithms, the logarithm of 125 to the base $4\frac{1}{2}$.

4. If $\phi(n)$ denote n^3+3n^2+3n, find $\phi(n+1)$ and $\phi(-1)$.

5. Find the number of arrangements of the letters of the product of a^7b^3 so that all the b's come together.

6. Find the scale in which the common number 3963 becomes 30203.

7. Find the coefficient of x^5 in the product

$$(1-x)(1+nx+\frac{n\,.\,\overline{n+1}}{\lfloor 2}x^2+\frac{n\,.\,\overline{n+1}\,.\,\overline{n+2}}{\lfloor 3}x^3+\ldots \textit{ad inf.}).$$

8. If $y=x+x^2+x^3+x^4+\ldots$ *ad inf.*, prove that

$$x=y-y^2+y^3-y^4+\ldots \textit{ad inf.}$$

9. Sum to n terms the series whose n^{th} term is $8n^3-1$.

10. Prove $\log_e 3 = 1 + \frac{1}{3\,.\,2^2} + \frac{1}{5\,.\,2^4} + \frac{1}{7\,.\,2^6} + \ldots$ *ad inf.*

LIV. aa.

1. If $\phi(x) = 1 - 3x + 2x^2$, prove that $\phi(x) \times \phi(-x) = 2\phi(x^2) - 1 + x^2$.

2. The numbers 104020 and 20 are in the scale of 9. Divide the first by the second and express the result in the common scale.

3. Show that $3(1+3+5+\ldots+99) = 101+103+105+\ldots+199$.

4. A and B together can do a piece of work in a certain time. If they each did half the work separately, A would have to work one day less, B 2 days more than before. Find the time in which A and B together do the work.

5. ABCDEFG is a regular octagon. Prove that the lengths of AB, AC, AD are proportional to 1, $\sqrt{2+\sqrt{2}}$, $1+\sqrt{2}$.

6. If $x = a+b+\frac{(a-b)^2}{4(a+b)}$, and $y = \frac{a+b}{4} + \frac{ab}{a+b}$, show that
$$(x-a)^2 - (y-b)^2 = b^2.$$

7. Find 2 numbers such that their sum, their difference, and the sum of their squares are proportional to 7, 1, 75.

8. The side BC of a $\triangle$ ABC is produced to P, where $BC = a$ and $CP = x$. Compare the areas of the $\triangle^s$ ABC, ABP.

The sides BC, CA, AB of a $\triangle$ ABC are produced to P, Q, R, where $CP = AQ = BR = x$. If $BC = a$, $CA = b$, $AB = c$, show that the ratio of the $\triangle$ PQR to the $\triangle$ ABC is
$$[(a+x)(b+x)(c+x) - x^3] : abc.$$

9. To do a piece of work a contractor can employ two classes of workmen, whose wages are in the ratio of 17 to 13. If he employs the higher paid men (who work the faster), he pays £148. 15*s.* in wages, being £10. 10*s.* less than the lower paid men would cost him. Compare their rates of work.

10. A dam has to be built of material q times as heavy as water; and its thickness in feet T at any depth d feet is calculated from the formula $T = q^{-\frac{1}{2}}d$. When the material is 2·3 times as heavy as water, what must the thickness be at a depth of 8 feet?

LIV. bb.

1. Find the value of $\left(\frac{x+a}{x-a}\right)^2 - \frac{x}{3a}$ when $x = a(1+2\sqrt{3})$.

2. Simplify $\sqrt{(6-\sqrt{17-12\sqrt{2}})}$.

3. Transform 123456 from scale 10 to scale 7

4. Find $\sqrt[5]{242}$ correct to 5 places by the Binomial Theorem.

5. The general term of $(1-4x)^{-\frac{3}{2}}$ is $\frac{\lfloor 2r+1}{(\lfloor r)^2}x^r$.

6. Expand $\frac{5-10x}{2-x-3x^2}$ and give the general term.

7. Solve the equation $3^{2x} \cdot 5^{3x-4} = 7^{x-1} \cdot 11^{2-x}$.

8. Find the values of x and y which satisfy the equations $y = x+2$, $y = bx+3$. Drawing graphs of the equations on squared paper, indicate the values of x and y which satisfy both equations (i) when $b = 0{\cdot}7$; (ii) when $b = 0{\cdot}8$; (iii) when $b = 0{\cdot}9$. Is it possible to find values of x and y which satisfy the two equations when $b = 1$?

9. The height h of the eye above the sea, and the distance d of the horizon, are connected by the equation $h^2 + 2h\mathsf{R} = d^2$, where R is the earth's radius. Taking $\mathsf{R} = 3960$ miles, find, in feet, the height above the sea at which the distance of the horizon is 5 miles.

10. The population of Ireland was 4,458,775 at the census of 1901, and decreased so that in x years from the census it fell to $(0{\cdot}9477)^x$ of the population at the census. Estimate the population $2\frac{1}{4}$ years after the census to the nearest thousand.

LIV. cc.

1. Show that $2(ab+cd)$ cannot be greater than $a^2+b^2+c^2+d^2$.

2. Find, by logarithms, the cube root of 3·2 and that of $\frac{2}{3}$.

3. Find the range of values of a for which the expression

$$(a-2)x^2 + 2(2a-3)x + 5a - 6$$

may have real factors.

4. Two men are walking with the same speed in opposite directions along a path parallel to a railway; a train passes one of them in a time t and the other in a time T; show that the train will describe a distance equal to its own length in a time which is the harmonic mean between T and t.

5. Find the product of $1 + x + x^2 + \ldots + x^{2n-1}$

and $1 - x + x^2 - \ldots - x^{2n-1}$.

6. Find within what limits the values of x must be when $x^2 - 7x + 12$ is negative. Verify your result.

7. If $a^2 + b^2 = 11ab$, show that

$$\log \frac{a-b}{3} = \frac{1}{2}(\log a + \log b).$$

8. An examiner has given marks to papers; the highest number of marks is 185, the lowest 42. He desires to change all his marks according to a linear law converting the highest number of marks into 250 and the lowest into 100; show how he may do this, and state the converted marks for papers already marked 60, 100, 150. Use squared paper, or algebra, as you please.

9. A is the horizontal sectional area of a vessel in square feet at the water level, h being the vertical draught in feet.

A	14,850	14,400	13,780	13,150
h	23·6	20·35	17·1	14·6

Plot on squared paper and read off and tabulate A for values of h, 23, 20, 16.

If the vessel changes in draught from 20·5 to 19·5, what is the diminution of its displacement in cubic feet?

10. Work the following three exercises as if in each case one were alone given, taking in each case the simplest supposition which your information permits :

(i) The total yearly expense in keeping a school of 100 boys is £2,100 ; what is the expense when the number of boys is 175 ?

(ii) The expense is £2,100 for 100 boys, £3,050 for 200 boys ; what is it for 175 boys ?

(iii) The expenses for three cases are known as follows :

£2,100 for 100 boys, £2,650 for 150 boys, £3,050 for 200 boys. What is the probable expense for 175 boys ?

If you use a squared paper method, show all three solutions together.

LIV. dd.

1. If pv^k is constant ; and if $p=1$ when $v=1$, find for what value of v, p is 0·2. Do this for the following values of k, 0·8, 0·9, 1·0, 1·1. Tabulate your answers.

2. Solve the equation $(x-2)(x-4)=n$, and write down four values of n, for each of which at least one of the values of x will be a positive whole number.

3. Find the range of values of a, for which both the roots of the equation $x^2-2ax+a^2-1=0$ are less than 4 and greater than -2.

4. A certain quantity y equals the sum of two quantities, one of which varies directly as x, and the other inversely as x ; given that $y=2$ when $x=-1$, and that $y=4$ when $x=2$, find y in terms of x.

5. Show by the Binomial Theorem that $\left(\frac{9}{10}\right)^{\frac{4}{5}}=0{\cdot}919166$, and verify the result by a logarithmic calculation. If you find a small difference between the two results, what explanation (of a general kind) can be given of the difference ?

6. Compute $2{\cdot}307^{0{\cdot}65}$ and $23{\cdot}07^{-1{\cdot}25}$, using logarithms.

7. If $w=144\{p_1(1+\log r)-r(p_3+10)\}$, and if $p_1=100$, $p_3=17$; find w for the four values of r, $1\frac{1}{2}$, 2, 3, 4. Tabulate your answers.

8. Express $\dfrac{x-13}{x^2-2x-15}$ as the sum of two simpler fractions.

9. Suppose s the distance in feet passed through by a body in the time t seconds is $10t^2$. Find s when t is 2, when t is 2·01, and when $t=2{\cdot}001$. What is the average speed in each of the two short intervals of time after $t=2$? When the interval of time is made shorter and shorter, what does the average speed approximate to ?

10. $z=ax-by^3x^{\frac{1}{2}}$.

If $z=1{\cdot}32$ when $x=1$, and $y=2$;
and $z=8{\cdot}58$ when $x=4$, and $y=1$; find a and b.

Then find z when $x=2$ and $y=0$

LIV. ee.

1. If n be a positive number, show that $\dfrac{1}{\sqrt{(n^2+1)}-n}$ is greater than $2n$.

2. From the logarithm tables find the value of the fifth root of 1·125.

3. If x is a positive number less than unity, find for what values of x the product $x(1-x)$ is less than $\frac{1}{6}$. With the same restriction on the values of x, if we write n instead of 6, show that n must be greater than 4.

4. Without using a table of logarithms, find the characteristic in the following cases:

(i) of the common logarithm of $\frac{1}{1\cdot 2^5}$.

(ii) ..... $\sqrt[3]{\frac{3}{7}}$.

(iii) of the logarithm to the base 3 of 776.

(iv) ..... 0·0776.

5. Compute, using logarithms,

$$\sqrt[3]{37\cdot 24},\ \sqrt{3\cdot 724},\ 372\cdot 4^{2\cdot 43},\ 0\cdot 3724^{-2\cdot 43}.$$

Find the third result to the nearest thousand.

6. Find within what limits the value of x must be when $\dfrac{x-1}{6-x}$ is positive. Verify your result.

7. A firm is satisfied from its past experience and study that its expenditure per week in pounds is

$$120+3\cdot 2x+\frac{C}{x+5}+0\cdot 01C,$$

where x is the number of horses employed by the firm and C the usual turnover.

If C is 2150 pounds, find for various values of x what is the weekly expenditure, and plot on squared paper to find the number of horses which will cause the expenditure to be a minimum.

8. If $A=P\left(1+\dfrac{r}{100}\right)^n$ and if $A=3P$ when $r=3\frac{1}{2}$, find n.

9. The New Zealand Pension law for a person who has already lived from the age of 40 to 65 in the colony is:

If the private income I is not more than £34 a year, the pension P is £18 a year. If the private income is anything from 34 to 52, the pension is such that the total income is just made up to 52. If the private income is 52 or more there is no pension.

Show on squared paper, for any income I, the value of P, and also the value of the total income. If a person's private income is say £50, how much of it has he an inducement to give away before he applies for a pension?

Show on the same paper the total income, if the pension were regulated according to the rule

$$P=18-\frac{9}{26}\cdot I.$$

10. Express $\frac{3x-2}{x^2-3x-4}$ as the sum of two simple fractions.

11. The following are the areas of cross section of a body at right angles to its straight axis:

A in square inches -	250	292	310	273	215	180	135	120
x inches from one end -	0	22	41	70	84	102	130	145

Plot A and x on squared paper. What is the probable cross section at $x=50$? What is the average cross section and the whole volume?

12. The following table records the heights in inches of a girl A (born Jan. 1890) and a boy B (born May 1894). Plot these records. The intervals of time may be taken as exactly 4 months.

Year - -	1900	1901			1902			1903
Month - -	Sept.	Jan.	May	Sept.	Jan.	May	Sept.	Jan.
A - - -	54·8	55·6	56·6	58·0	59·2	60·2	60·9	61·3
B - - -	48·3	49·0	49·8	50·6	51·5	52·3	53·1	53·9

Find in inches per year the *average* rates of growth of A and B during the given period. At about what age was the growth of A most rapid? State this rate; divide it by her average rate.

APPENDIX.

GRAPHIC SOLUTION OF CUBIC EQUATIONS.

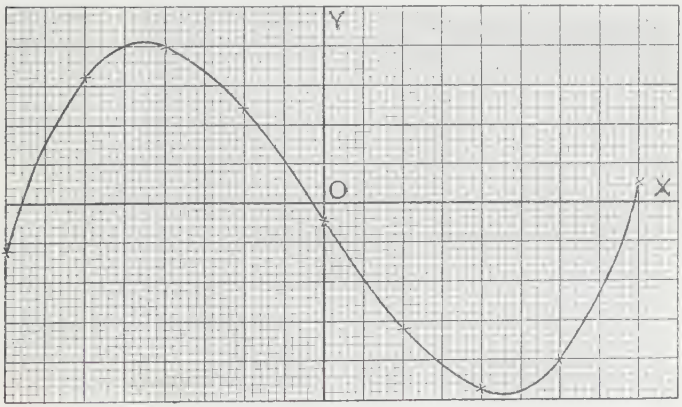
Solve the equation $x^3 - 15x - 2 = 0$.

Let $y = x^3 - 15x - 2$.

$x =$	-4	-3	-2	-1	0	1	2	3	4
$x^3 =$	-64	-27	-8	-1	0	1	8	27	64
$-15x =$	60	45	30	15	0	-15	-30	-45	-60
$-2 =$	-2	-2	-2	-2	-2	-2	-2	-2	-2
$y =$	-6	16	20	12	-2	-16	-24	-20	2

Use 1 inch as x unit, one-tenth of an inch as y unit.

Plotting these points and joining them by an even curve, we have the graph as shown in the diagram.



Where it cuts the axis of x,

$$x = -3{\cdot}82, \quad -0{\cdot}13, \quad 3{\cdot}95,$$

which are the required roots.

1. Plot the curve $y=x^3-6x+5$, and hence solve the equation

$$x^3-6x+5=0.$$

2. Plot the curve $y=x^3+2x^2-5x-6$, and hence, solve the equation

$$x^3+2x^2-5x-6=0.$$

3. Plot the curve $y=x^3+6x-7$ for the following values of x,

$$-4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4.$$

What do you deduce as to the roots of the equation $x^3+6x-7=0$?

4. Solve the equation $x^3-10x=7$.

5. Plot the curve $y=x^3-3x^2-10x+20$ for the following values of x, $-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4$, and hence obtain two roots of the equation. Deduce the value of the third root.

6. Plot the curves $y=x^3$ and $y=x^2-5$, and hence solve the equation $x^3-x^2+5=0$. What do you deduce as to the other roots?

7. Using as large units as convenient, plot in the same diagram between the points (0, 0), (1, 1) the graphs of $y=x^n$ when $n=3,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{3}$.

ANSWERS TO THE EXAMPLES.

PART II.

XXXV. a. (p. 286).

1. $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$.
2. $x^6-ax^5+a^2x^4-a^3x^3+a^4x^2-a^5x+a^6$.
3. $x^6+ax^5+a^2x^4+a^3x^3+a^4x^2+a^5x+a^6$.
4. $x^7-ax^6+a^2x^5-a^3x^4+a^4x^3-a^5x^2+a^6x-a^7$.
5. $x^7+ax^6+a^2x^5+a^3x^4+a^4x^3+a^5x^2+a^6x+a^7$.
6. $x^5+2x^4+4x^3+8x^2+16x+32$.
7. $x^5-2x^4+4x^3-8x^2+16x-32$.
8. $(x+1)(x-2)(x+2)$.
9. $(x^2+x-3)(x^2+x+1)$.
10. $\left(\frac{3}{x}-1\right)\left(\frac{9}{x^2}-\frac{3}{x}+1\right)$.
11. $\left(a+\frac{1}{a}\right)\left(a^2-1+\frac{1}{a^2}\right)$.
12. $(x-2)^2(x-1)(x-3)$.
13. $(x+1)(x-1)(y+1)(y-1)$.
14. $(c-a)(a-b+c)$.
15. $(x-1-\sqrt{3})(x-1+\sqrt{3})$.
16. $(a^2+5a+1)(a^2-5a+1)$.
17. $(ax-1)(bx^2+ax+1)$.
18. $(a-b)(a+b+c)$.
19. $(a+b+1)(a^2+b^2+1-a-b-ab)$.
20. $(x-5-\sqrt{2})(x-5+\sqrt{2})$.
21. $(2x-y-8)(2x-y+3)$.
22. $(x^2+2x+2)(x^2-2x+2)$.
23. $(x-2y-3)(x^2+y^2+9-6y+3x+2xy)$.
24. $(2x-2-\sqrt{3})(2x-2+\sqrt{3})$.
25. $(3x-4y+2)(3x+4y-5)$.
26. $(5x-2y+7)(3x-2y+4)$.
27. $(x^2+x+1)(x^2-x+1)$.
28. $(a-b+2)(a^2+b^2+4+ab-2a+2b)$.
29. $(4x+1-\sqrt{6})(4x+1+\sqrt{6})$.
30. $(4x^2+2x+1)(4x^2-2x+1)$.
31. $(2a+b-1)(4a^2+b^2+1-2ab+b+2a)$.
32. $(2x-1)(x-2)(2x^2-5x+5)$.
33. $(3x-5y+7)(2x-3y+3)$.
34. $(x^2+4x+8)(x^2-4x+8)$.
35. $(x^2+3xy+4y^2)(x^2-3xy+4y^2)$.
36. $(x+1)^2(x^2+2x-11)$.
37. $(x+a)(x^2+ax+a^2)(x^2-ax+a^2)$.
38. $(9a^2-3a+1)(9a^2+3a+1)$.
39. $(2x+3y-2)(x+y+4)$.
40. $(x^2+2x+4)(x^2-2x+4)$.
41. $(9x^2+12x+8)(9x^2-12x+8)$.
42. $(x-a)(x^2+ax+a^2)(x^2-ax+a^2)$.
43. $\left(x+4+\frac{1}{x}\right)\left(x-3+\frac{1}{x}\right)$.
44. $(x-1)(x-2)(x-3)$.
45. $\left(x+2+\frac{2}{x}\right)\left(x-7+\frac{2}{x}\right)$.
46. $(2x-1)(x+1)(x+2)$.
47. $\left(x+\frac{1}{x}\right)\left(x^2-6+\frac{1}{x^2}\right)$.
48. $\left(x-\frac{1}{x}\right)^2\left(x^2+5+\frac{1}{x^2}\right)$.

49. $x^2+4xy+y^2-9z^2$.
50. $4x^2-9y^2+16z^2+16xz$.
51. $a^2+2ab+b^2-5a-5b+6$.
52. $a^2+2ab+b^2-x^2-2xy-y^2$.
53. x^4+x^2+1.
54. x^4+3x^2+4.
55. x^4-x^2-4x-4.
56. $x^4+x^2y^2+25y^4$.
57. $4a^2+4ab+b^2-x^2-4xy-4y^2$.
58. $4x^4-9x^2y^2+24xy^3-16y^4$.
59. $x^6+2x^5+x^4-x^2-2x-1$.
60. x^4+4x^2+16.
61. $9(a^2-b^2)^2$.
62. $x^2+1+\frac{1}{x^2}$.
63. $x^6-2x^4-7x^2-4$.
64. $32x^6-32x^2$.
65. $6a^6x^2-6a^2x^6$.
66. a^6-b^6.
67. $a^8-2a^4b^4+b^8$.
68. $225x^4-244x^2+64$.
69. $36x^4-73x^2+16$.
70. x^6-64.
71. $a^{12}-3a^8b^4+3a^4b^8-b^{12}$.
72. $64a^8-128a^4b^4+64b^8$.
73. $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$.
74. $5x^2+x+6$.
75. $4x^2-1$.
77. $3x^2+7x+2$.
78. $16-8b$.
80. x^2-b^2.
81. $81a^4-72a^2b^2+16b^4$.
82. $x^3+x^2y+xy^2+y^3$.
83. $x^3-x^2y+xy^2-y^3$.
84. $x^2+2x^2y+4xy^2+8y^3$.
85. $x^3-2x^2y+4xy^2-8y^3$.

XXXV. b. (p. 294).

6. $3(a-b)(b-c)(c-a)$.
7. $(a-b)(b-c)(c-a)(ab+bc+ca)$.
8. $-(x-y)(x+y)(y-z)(y+z)(z-x)(z+x)$.
9. $-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.
19 0.
20. $-\frac{1}{abc}$.
21. 1.
22. $(a-b)(b-c)(c-a)$.
23. 0.
24. $\frac{1}{xyz}$.
25. $\frac{a^2}{(a-x)(a-y)(a-z)}$.
26. $\frac{3(b-c)(c-a)(a-b)}{(a+b)(b+c)(c+a)}$.
27. $ab+bc+ca$.
28. 0.
29. $\frac{1}{abc}$.
30. 0.
31. $\frac{1}{a^2b^2c^2}$.
32. 1.
33. $-(a+b+c)$.
34. $\frac{1}{(x-a)(x-b)(x-c)}$.
35. $\frac{x}{(x-a)(x-b)(x-c)}$.
36. $\frac{x^2}{(x+a)(x+b)(x+c)}$.
37. -1.
38. 0.
39. 1.

XXXV. c. (p. 298).

1. $p=16a^2$.
4. 4.
6. $pq+r=0$.
11. $x=a$, $y=b$.
14. $p^2+bp+ac=0$.
15. $x=-b$, $y=a$.
16. 3.
17. $8x+4$.
21. $x=\frac{1}{2}$, $y=1$.
23. 3.
26. $(2x^3-xy^2+3y)^2$.
27. $1\frac{1}{3}$.
28. 0.
29. $-(yz+zx+xy)$.

XXXVI. a. (p. 300).

1. $\sqrt{18}$, $\sqrt{50}$, $\sqrt{45}$. **2.** $\sqrt{147}$, $\sqrt{18}$, $\sqrt{20}$.
3. $2\sqrt{3}$, $2\sqrt{2}$, $4\sqrt{2}$, $5\sqrt{3}$. **4.** $7\sqrt{5}$, $9\sqrt{3}$, $15\sqrt{2}$, $5\sqrt{7}$.
5. $2\sqrt[3]{2}$, $2\sqrt[3]{4}$, $10\sqrt{5}$, $13\sqrt{3}$. **6.** $14\sqrt{2}$, $27\sqrt{3}$, $9\sqrt[3]{3}$, $-6\sqrt[3]{3}$.
7. $25\sqrt{2}$, $3\sqrt[3]{3}$, $7\sqrt[3]{3}$, $8\sqrt[3]{2}$, $-5\sqrt[3]{2}$. **8.** $4\sqrt[3]{5}$, $2\sqrt[5]{2}$, $5\sqrt[3]{3}$, $3\sqrt[5]{2}$, $ab\sqrt[5]{a^2b^3}$.
9. $ab^2c^2\sqrt{ac}$, $ab^2\sqrt[3]{ab}$, $(a-b)\sqrt{b}$. **10.** $41\sqrt{3}$. **11.** $7\sqrt{5}+17\sqrt{2}$.
12. $-4\sqrt{6}$. **13.** $-7\sqrt{5}$. **14.** $5\sqrt[3]{3}$.
15. $28\sqrt[3]{4}$. **16.** $29-2\sqrt{6}$.
17. $42+\sqrt{105}-6\sqrt{21}-3\sqrt{5}$. **18.** 87. **19.** 33.

XXXVI. b. (p. 302).

1. $2\sqrt{3}$. **2.** $\sqrt{5}$. **3.** $\frac{1}{4}\sqrt{14}$.
4. 3·15. **5.** 1·41. **6.** 1·73.
7. 4·24. **8.** 1·44. **9.** 1·79
10. 1·32. **11.** 2·83. **12.** 4.
13. 11·3137. **14.** $3\sqrt{7}$. **15.** Difference 1·46.
16. $2\sqrt{15}-7$. **17.** $\frac{1}{17}(2\sqrt{5}-\sqrt{3})$; $\frac{1}{13}(24-\sqrt{30})$.
18. $17+4\sqrt{15}$; $65-6\sqrt{14}$; $68+16\sqrt{15}$; $10+2\sqrt{15}+2\sqrt{10}+2\sqrt{6}$.
19. $5\sqrt{3}+3\sqrt{2}$; 2. **20.** $\sqrt{5}+1$. **21.** 1·41.
22. 1·73. **23.** 2·65, 2·83, 2·24. **24.** 3·32.
25. 3·61. **26.** 4·58. **27.** 4·38. **28.** 3·15.
30. $3-2\sqrt{2}$. **31.** $\sqrt{5}-1$. **32.** $\sqrt{5}-\sqrt{2}$. **33.** 2.
34. $\frac{1}{6}(27+11\sqrt{6})$. **35.** $\dfrac{a-\sqrt{a^2-b^2}}{b}$. **36.** $\dfrac{(2x-y)\sqrt{x+y}+(2x+y)\sqrt{x-y}}{y^2}$.
37. ·81. **38.** 23·31. **39.** 1·42. **40.** 1·20.

XXXVI. c. (p. 303).

1. $\sqrt{3}+1$. **2.** $\sqrt{6}+1$. **3.** $3-\sqrt{3}$. **4.** $3+\sqrt{2}$.
5. $2\sqrt{7}+\sqrt{2}$. **6.** $3-2\sqrt{2}$. **7.** $\sqrt{7}+\sqrt{5}$. **8.** $2\sqrt{5}-2\sqrt{3}$.
9. $\dfrac{\sqrt{5}}{2}-1$. **10.** $2\sqrt{5}+\sqrt{7}$. **11.** $2\sqrt{13}-7$. **12.** $\sqrt{2}+\frac{1}{6}$.
13. $\sqrt[4]{2}\{\sqrt{3}-1\}$. **14.** $3\sqrt{3}-\sqrt{6}$. **15.** 2·732. **16.** 3·449.
17. 1·268. **18.** 4·414. **19.** 6·706. **20.** ·172.
21. 4·882. **22.** 1·008. **23.** ·118. **24.** 7·118.
25. ·212. **26.** 1·581. **27.** ·871. **28.** 2·747.
29. $\sqrt{3}+\sqrt{2}$. **30.** $\sqrt{2}+1$. **31.** $\sqrt{5}-1$. **32.** $\sqrt{7}-\sqrt{5}$. **33.** ·3090.

XXXVI. d. (p. 304).

1. $2\sqrt{2}$, $10\sqrt{2}$, $2\sqrt[3]{2}$, $2\sqrt{14}$, $5\sqrt[3]{2}$, $\sqrt{6}$, $2\sqrt{7}$, $2\sqrt{3}$, $5\sqrt{2}$, $5\sqrt{3}$.
2. 20, 90, 48, 28, 125, $4+2\sqrt{3}$, $5+2\sqrt{6}$, $6-2\sqrt{5}$, $9+6\sqrt{2}$, $9-6\sqrt{2}$.
3. $2\sqrt{3}+\sqrt{7}$, $5\sqrt{3}-2\sqrt{5}$, $4+\sqrt{15}$, $\sqrt{5}-1$, $5+2\sqrt{2}$, $2\sqrt{3}+\sqrt{2}$.

XXXVI. e. (p. 304).

1. $\sqrt{5}+2\sqrt{3}$. **2.** 5. **3.** $2\sqrt{5}=4{\cdot}472$.

4. $\frac{1}{3}\sqrt{6}={\cdot}816$. **5.** 1. **6.** $\sqrt{2}+1$.

7. $\frac{\sqrt{5}}{30}(9\sqrt{5}-11\sqrt{2})$; $\frac{\sqrt{3}}{18}(9\sqrt{5}-11\sqrt{2})$; ·117. **8.** 50.

9. 1, or $\frac{c^2}{b^2}$. **10.** $1+\sqrt{2}+\sqrt{3}$; $\sqrt{2(1+a)}$. **11.** 1.

XXXVII. a. (p. 308).

1. $a^{\frac{1}{3}}$. **2.** $a^{\frac{2}{5}}b^{\frac{1}{5}}$. **3.** $3b^{\frac{1}{2}}c^{-\frac{1}{3}}$. **4.** $a^{\frac{1}{3}0}$; $x^{-\frac{1}{2}}y^{-\frac{1}{4}}$; $2^{\frac{1}{2}}a^{-\frac{3}{2}}$.

5. 4; $\frac{1}{5}$. **6.** $\frac{1}{9}$; 343. **7.** $\frac{bc}{a}+\frac{c}{a^2b}+\frac{a}{bc}$. **8.** 8. **9.** $\frac{1}{8}$.

10. $\frac{1}{17}$. **11.** $\frac{1}{8}$. **12.** $\frac{1}{81}$. **13.** 125. **14.** 200. **15.** 49.

16. $10y^{\frac{3}{2}}+29x^{\frac{1}{4}}y+16x^{\frac{1}{2}}y^{\frac{1}{4}}-7x^{\frac{3}{4}}$. **17.** x^3+1+x^{-3}.

18. $a^{-2}+a^{-1}b^{-1}+b^{-2}$. **19.** a^3-b^2.

20. $a^2+4ab^{\frac{1}{4}}+4b^{\frac{1}{2}}-9c$. **21.** $x-y^4$.

22. $a^4-9a^2b^{-2}-24ab^{-1}c^{-2}-16c^{-4}$. **23.** $4x^2+6xy^{-1}+9y^{-2}$.

24. $x^{-\frac{5}{2}}-2x^{-2}y^{\frac{1}{3}}+4x^{-\frac{3}{2}}y^{\frac{2}{3}}-8x^{-1}y+16x^{-\frac{1}{2}}y^{\frac{4}{3}}-32y^{\frac{5}{3}}$.

25. $\left(a^{\frac{1}{4}}+b^{\frac{1}{4}}\right)^2$. **26.** $a^{\frac{1}{4}}b^{-\frac{1}{4}}+1+a^{-\frac{1}{4}}b^{\frac{1}{4}}$.

27. $3x^{\frac{1}{3}}+5y^{\frac{1}{2}}-2z$. **29.** $e^{\frac{1}{2}}=1{\cdot}65$.

30. $7^{\frac{1}{2}}=2{\cdot}645751131$; $7^{\frac{1}{4}}=1{\cdot}6265766$; $7^{\frac{1}{8}}=1{\cdot}275373$; $7^{\frac{1}{16}}=1{\cdot}12935$; $7^{\frac{1}{32}}=1{\cdot}0627$; $x=8$; 4·303, 1·838.

31. 1·015. **32.** $100-99{\cdot}47$.

XXXVII. b. (p. 309).

1. $\frac{x^6}{a}$. **2.** $\frac{a^3}{2b^2}$. **3.** $\frac{3c}{b^2}$. **4.** $\frac{x^6}{64y^4}$. **5.** $\frac{2b^2}{a}$.

6. $\frac{ab}{b-a}$. **7.** $a-\sqrt{ab}+b$. **8.** $\frac{1}{16}$. **9.** 256. **10.** $\frac{2}{5}$.

XXXVII. c. (p. 309).

1. 8, −1, 0. **2.** $\sqrt[3]{2}$. **3.** $\sqrt[5]{a^{-2}}-\frac{1}{4}\sqrt{b}$. **4.** $b^{\frac{7}{5}}c^{-\frac{3}{5}}-b^{\frac{4}{5}}-c^{\frac{4}{5}}$.

5. $a^{\frac{5}{2}}+a^2b^{-\frac{1}{2}}+a^{\frac{3}{2}}b^{-1}+ab^{-\frac{3}{2}}+a^{\frac{1}{2}}b^{-2}+b^{-\frac{5}{2}}$. **6.** $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.

7. $\frac{1}{4}\left(a^{\frac{1}{2}}+a^{-\frac{1}{2}}\right)(5a-4+5a^{-1})$. **8.** $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

9. $\frac{x^{\frac{2}{3}}-2y^{-\frac{2}{3}}}{x^{\frac{2}{3}}+2y^{-\frac{2}{3}}}$, *i.e.* $\frac{(xy)^{\frac{2}{3}}-2}{(xy)^{\frac{2}{3}}+2}$. **10.** $x^6+3x^2+3x^{-2}+x^{-6}$.

XXXVIII. a. (p. 314).

4. 2 : 3. **5.** -7. **6.** 4 : 7. **7.** 5 : 8. **8.** 4.

9. 3 : 1 or 1 : 3. **10.** 10. **14.** $\frac{ad-bc}{c-d}$. **16.** $\frac{43}{59}$.

18. 80·1 feet.

XXXVIII. b. (p. 315).

1. $\frac{3}{5}$ greatest, $\frac{6}{11}$ least. **2.** ·6, ·56, ·54. **5.** £12 10*s.*, £7 10*s.*

6. 45°. **7.** 15. **8.** $\frac{18}{24}$, $\frac{20}{24}$, $\frac{21}{24}$.

9. 17. **10.** 1 : 2.

XXXVIII. c. (p. 316).

2. 1 inch represents 3 feet. **3.** 1 : 1296. **9.** $\frac{1}{2}$. **10.** $\frac{205}{316}$.

11. $\frac{1}{3}$. **15.** 40, 45. **16.** 32, 60. **17.** 31·25, 33·75.

23. 40, 16. **24.** 5 : 2. **26.** 19 : 3.

27. Scale $\frac{1}{17280}$, $\frac{121}{90}$ sq. in. **28.** 5 : 2 or 2 : 1. **29.** 11 : 24.

30. 136. **32.** $3\frac{1}{3}$ miles. **34.** 82, 65, 57.

XXXVIII. d. (p. 319).

1. $a : c = d : b$, or $a : d = c : b$. **2.** 21; $\frac{25x^5}{2}$. **3.** bc; $12yz$; $\frac{b^3c^3}{a^2}$.

9. 4, 3. **10.** 5, 15, 45. **11.** $\frac{5+3\sqrt{3}}{2}$.

12. $2(\sqrt{5}+\sqrt{2})$. **18.** $\frac{a+b}{x+y}=\frac{x+y}{a-b}$. **19.** $a\frac{\sqrt{5}-1}{2}$.

21. 2·7; 8·49. **22.** 341, 68, 45. **23.** 3, 12.

24. $a : c$. **26.** $\frac{2p}{a}$.

XXXVIII. e. (p. 322).

1. $5x=6y$; 7·5, 4·2. **2.** $\frac{2}{7}$. **3.** 101·25.

4. 24. **5.** $y=4x+3$; 17. **9.** $\frac{4}{9}$ lb.

10. $xy=\frac{12}{25}(x^2+y^2)$. **11.** $y=\frac{14}{4-5x}$. **15.** 36.

16. £200. **17.** 12 oz. **18.** £688. 10*s.* 8*d.* **19.** 20 miles an hour.

XXXVIII. f. (p. 323).

1. £1470. **2.** 78·5 sq. ms.; 2·45 ms. **3.** 336 feet.

4. £14. 4*s.* 5*d.* **5.** 44·5; $\frac{88}{37}$. **6.** 3 miles; $10\frac{2}{3}$ feet.

7. 1·074 secs. **8.** $4000\times\frac{81}{1681}$ miles = 193 miles, nearly. **9.** £34.

11. Expense $=(12+\frac{3}{4}x)$£.

XXXIX. a. (p 327).

1. 3, 5, 0, −2, −4. **2.** 6, 8, 0, −1, −4. **3.** 0.
4. 1·8060, ·5050, ·1070, $\bar{1}$·6990. **5.** 2·3855, 1·3855, 2·9084, $\bar{2}$·9084.
6. ·6990, 2·6990, $\bar{1}$·6020, 2·0970. **7.** 2·7602, 3·2373, 1·9363.
8. 2, 3, 1, $\bar{1}$, $\bar{2}$. **9.** 1·316. **10.** 5. **11.** ·002.

XXXIX. b. (p. 327).

1. 2, 2, 3, 3, 3. **2.** 3, 5, 3, 3, 6, 4.
3. $\frac{1}{2}$, $\frac{1}{3}$, 5, 5, $\frac{1}{2}$, $\frac{1}{3}$, **4.** $\frac{4}{3}$, $\frac{5}{2}$, $\frac{3}{4}$, $\frac{1}{3}$, −3, $-\frac{1}{2}$.
5. $\log a+\log b+\log c$, $2\log a+3\log b+5\log c$, $5\log x+\log y+2\log z$, $\log a+2\log b-\log c-2\log d$.
6. $2\log 2$, $\log 2+\log 3$, $3\log 2$. $2\log 3$, $1+\log 3-\log 2$, $\log 2+2\log 3$.
7. 1, $\log 6$, $2\log 2$, $\log 3$, $\log 6$, $3+\log 3$. **8.** 8. **9.** 3. **10.** 5.
11. −1. **12.** −3. **13.** 3, 1, 2, 4, $\bar{3}$, $\bar{1}$, 0, $\bar{5}$, 1.

XXXIX. c. (p. 328).

3. $\log(a+b)+\log(a-b)$. **4.** ·9542, 1·8573.
5. 1·2068. **6.** $\bar{3}$·9356. **7.** $\bar{3}$·4042. **8.** 2·4925.
9. $\bar{8}$·3855. **10.** $\bar{5}$·3980. **11.** $\bar{1}$·8474. **12.** $\bar{1}$·5796.
13. $\bar{1}$·4615. **14.** 13. **15.** 3·4134, ·4134, 5·4134, $\bar{3}$·4134.
16. 7. **17.** 8. **18.** 2035, 2035, 176, 176, 1308.
19. $\bar{3}$, 0, $\bar{3}$. **20.** 3·2454, ·2454, $\bar{3}$·2454.
21. 5·6956. **22.** 1·73, 2·65, 1·58, 3·32.
24. 1·5741. **25.** 3·5474. **26.** 1·5474, 5·5474, $\bar{2}$·5474.
27. 26. **28.** Indices ·477, ·602, ·699, 2·021.

XXXIX. d. (p. 334).

1. 4·5663, 2·5663. **2.** 3·1951, $\bar{2}$·1951.
3. 2·6320, 6·6320. **4.** 3·3250, $\bar{3}$·3250.
5. 2·5872, $\bar{2}$·7568, 6·8832, $\bar{1}$·8836. **6.** 2·449.
7. 2·646. **8.** 1·913. **9.** 1·58. **10.** 1·47.
11. $x=3{\cdot}23$, $y=1{\cdot}52$. **12.** 5, 3·5, $\bar{2}$·5, 6.
13. 4, 3, 9. **14.** 2·8, 10·55, 22·6, 3·8, 4·5.
15. 2·4013, 1·4013, 4·4013, $\bar{1}$·4013, $\bar{3}$·4013. **16.** 2·5383, 2·7960, 2·3892.
17. 2·1505. **18.** $\log 7$. **19.** ·8, 5·27.
21. 2921. **22.** 1·46. **23.** 14·04. **24.** ·1379.
25. ·4651. **26.** 191·4. **27.** ·0984. **28.** 41·04.
29. ·076. **30.** ·00052. **31.** ·0357. **32.** ·00231.
33. 13·57. **34.** ·9434. **35.** ·000016. **36.** 3·162.
37. 54·88. **38.** 41. **39.** 3·74. **40.** 7·14.
41. ·8899. **42.** 1·240, 1·433, ·176, ·30. **43.** 31·65.
45. x. **47.** ·7927. **50.** 5·82. **51.** 0
52. ·2345. **53.** 2·079. **54.** (i) 105·5, (ii) 849·4.

XL. a. (p. 343).

1. 2·4 hrs. **2.** 2·5 hrs. **3.** $\frac{3}{5}$.
4. 6·7 days. **5.** 3 days. **6.** 4 a.m.
7. 4, 8, 12, 16, 20 min. ; 352, 704, 1056, 1408 yds. from the start.
8. 12, 24, 36 min. ; ·4, ·8, ·2 miles. **9.** £27, 1330 marks. **10.** 2·84.
11. 9 yds. **12.** 7 miles an hour. **13.** 2*s.* 10*d.* **14.** 24 min.
15. 800 yds. **16.** 2400 feet from start. **17.** 11. **18.** 7.
19. 30 miles. **20.** 1 hr. 20 min. **21.** 5·7 miles.
22. 5·14 miles. **23.** $2\frac{3}{4}$ miles.
24. Ran 2 miles, distance 3 miles. Meet 21·2 min. and 60 min. from start, 1·4 and 1·5 miles from first man's starting point.
25. C travels 9·3 miles an hour, meets B 23·2 miles from first end.
26. 33·6 miles from B's start, in 4·8 hrs.
27. 5·25 hrs. **28.** 1 hr.
29. (i) (1) 2.11. (2) 2.27. (3) 2.33.
(ii) (1) 6.33. (2) 6.16 and 6.49. (3) 6.11 and 6.55.
(iii) (1) 9.49. (2) 9.33. (3) 9.27.
(iv) (1) Never. (2) 11.42. (3) 11.33.
30. 4·05 days approx. **31.** 8, 24 days.
32. 8, 12 gallons an hr. **33.** 90, 180 days.
34. 70%, 30%. **35.** 11 : 4.
36. 21·4 hrs. approx. and 50 hrs. **37.** 2 hrs.
38. 9 miles an hour, 12 miles an hour.
39. 108000, 155520, 62500, 75000, 80000, 85000, 90000, 96000, 102000.
40. At 3.20, $13\frac{1}{9}$ miles from A.

XL. b. (p. 351).

1. 4 and 6 miles an hour. **2.** 6 and 4 hours.
3. 45 miles ; $1\frac{2}{3}$, $3\frac{1}{3}$ miles an hour.
4. 35 miles ; 3·8 and 1·2 miles an hour. **5.** $3\frac{3}{4}$, $2\frac{1}{2}$ hours.
6. 10, 4 hours. **7.** 16, 4 days. **8.** 25, 15 days.
9. A $2\frac{1}{2}$, B 2 miles an hour ; dist. 5 miles. **10.** 6 miles an hour.

XL. c. (p. 352).

1. 1020 yds. **2.** 80. **3.** 7200, 4800. **4.** £1000. **5.** 17, 4, or 20, 1.
6. 70 lbs. **7.** In 2 hrs. 32 min. after man leaves A, $8\frac{1}{1}\frac{3}{5}$ miles from A.
8. £12, £10. **9.** 22. **10.** 9. **11.** *bd* shillings. **12.** 72.
13. 2. **14.** 2 hrs. $55\frac{2}{7}\frac{5}{7}$ min. **15.** 1400.
16. $\frac{3}{5}$ pint from first, $\frac{2}{5}$ from second. **17.** 10, 12 min.
18. $\frac{7}{3}\frac{}{4}$, $\frac{2}{3}\frac{7}{4}$. **19.** 80. **20.** 1*s.* 4*d.*, 2*s.* 10*d.*
22. 13 innings, average 17. **23.** £5, £3. 17*s.* 6*d.* **24.** 20.
25. 159, 141. **26.** 6, 12, 36. **27.** 36 wheat, 42 barley. **28.** 39.

XLI. a. (p. 354).

1. $x^2-2x+3-2x^{-1}+x^{-2}$; $a+3a^{\frac{2}{3}}b^{\frac{1}{3}}+3a^{\frac{1}{3}}b^{\frac{2}{3}}+b$. **2.** 3·01.
5. 54 : 35 ; 44%. **6.** 17. **7.** $\frac{3}{2}$, -1, $-\frac{3}{2}$, 2.

XLI. b. (p. 354).

2. 1·442. **3.** 576 feet, 176 feet.
5. $22\sqrt{2}$. **6.** 2·7202, ·6444. **7.** $10d$.

XLI. c. (p. 355).

1. $\frac{3}{5}$. **2.** 5, 12, 13 in.
3. $\frac{5}{3}(2\sqrt{6}+\sqrt{3})$. **4.** $4a^{\frac{1}{3}}+6a^{\frac{1}{6}}x^{-\frac{1}{6}}+9x^{-\frac{1}{3}}$.
6. Greater in the former case ; (i) $\frac{\pi-2}{\pi}$, (ii) $\frac{4-\pi}{4}$.

XLI. d. (p. 355).

1. $(ac+bd)^2+(ad-bc)^2$, or $(ad+bc)^2+(ac-bd)^2$.
2. 2, 4. **3.** ·097, 1·36, ·8.
4. 2 gals. from A, 14 from B. **6.** $\frac{45\sqrt{2}}{4}$. **7.** 2·596.

XLI. e. (p. 356).

1. $3+\sqrt{5}=5\cdot236$.
2. $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$, $x^{\frac{3}{5}}+y^{\frac{3}{5}}+z^{\frac{3}{5}}-y^{\frac{1}{5}}z^{\frac{1}{5}}-z^{\frac{1}{5}}x^{\frac{1}{5}}-x^{\frac{1}{5}}y^{\frac{1}{5}}$.
3. ±4. **5.** 7, 24, 25 in. **7.** $1\pm\sqrt{2}=2\cdot41$, $-\cdot41$.

XLI. f. (p. 356).

1. 2, 18. **2.** £11. **3.** the latter.
5. $-\frac{1}{6}=-\cdot1667$. **6.** £9. 10*s*. 9*d*. **7.** 1·707.

XLI. g. (p. 356).

1. 2. **2.** $\frac{1}{a}$ or a. **3.** ·4128. **5.** $w=2x+6z^2$.
6. (i) $x=4$ or 5, $y=5$ or 4. (ii) $x=3$.
7. at 5·7 miles from G, and 2·67 from G.

XLI. h. (p. 357).

1. $\{(ac+bd)e+(ad-bc)f\}^2+\{ac+bd)f-(ad-bc)e\}^2$. **2.** 50 feet.
3. -1, $-\frac{1}{2}$, -2. **6.** 1·8. **7.** 1·9452.

XLI. i. (p. 357).

1. $\frac{47}{2}\sqrt{5}$. **2.** $a^{-2}-3a^{-1}b+\frac{b^2}{3}$. **3.** ·6990, ·0791, 3·8293.
4. 1·27. **5.** $2\frac{1}{2}$ gallons. **7.** (i) $-1\pm\sqrt{7}$. (ii) 0, $\pm\sqrt{3}$.

XLI. k. (p. 358).

1. $\triangle = 3$; 2 : 1. 2. $\frac{a}{3}$. 3. 1920 yds.; $5\frac{1}{2}$ min.
4. $\frac{p^2-2q}{q}$. 5. $3+\sqrt{7}$. 6. 1. 7. $qy : px$.

XLI. l. (p. 358).

1. £48. 2. $\sqrt{2}$. 5. $\frac{3}{4}$ or $\frac{4}{3}$. 7. 2·327.

XLI. m. (p. 359).

2. $\frac{2\sqrt{5}}{3}(\sqrt{5}-1)$. 3. $3\frac{1}{3}$, $6\frac{2}{3}$. 4. 7.
5. ·3010. 6. 2 : 1. 7. 32 : 25.

XLI. n. (p. 359).

1. ·618. 3. 5, 6, 7 or 8. 4. 70. 5. 3. 6. £187. 10*s*.
7. $\frac{d}{t}\cdot\frac{nc}{(m-n)c-md}$, $\frac{d}{t}\cdot\frac{n(c-d)}{(m-n)c-md}$ yds. a minute;
distance $=\frac{(m-n)c(c-d)}{(m-n)c-md}$.

XLI. p. (p. 359).

1. $\frac{by+bz+ax-az}{by+bz-ax+az}$. H.C.F. $b(x+y)$. 2. 5 : 12.
3. $\bar{1}$·8733. 5. $2\frac{1}{4}$ feet. 6. 14 miles.

XLI. q. (p. 360).

1. $\frac{2}{7}ab^{-1}-5+\frac{3}{4}a^{-1}b$. 3. 14 miles. 4. 2 : 1. 5. 1 : 3 = 5 : 15.
6. ·1, ·316, ·562, 1·778, 3·162, 5·623; 1·585, 1·995, ·178.
7. $3^{\frac{1}{8}} = 1{\cdot}147015\ldots$.

XLII. a. (p. 364).

1. 7. 2. 40. 3. 23. 4. 70. 5. 24.
6. 4. 7. 7. 8. 8. 9. 26. 10. 45.
11. 8. 12. 20. 13. −7. 14. 47. 15. −34.
16. $3\frac{1}{2}$. 17. 320. 18. −44. 19. $2n-1$.
20. $l = 14\frac{3}{4}$, $s = 94\frac{1}{2}$. 21. $9\frac{1}{4}$, $61\frac{1}{2}$. 22. −115, −2075.
23. 5·25, 57·5. 24. −2·3, −24·8. 25. 0. 26. 17.
27. 133. 28. 163·2. 29. $n\{(5a-b)+2n(a-3b)\}$. 30. $15x+225$.

XLII. b. (p. 364).

1. 75, 507. 2. 52, 297. 3. −27, −51. 4. −75, −438.
5. 5500. 6. $-\frac{3}{2}$. 7. 48. 8. 6400.
9. $21\frac{7}{12}$. 10. $-169\frac{1}{2}$. 11. 1560, 1521.

12. 17 hrs.; 68 miles. **13.** 90. **14.** 205. **15.** £7. 7*s.* 6*d.*

16. $a-2d,\ a-d,\ a,\ a+d,\ a+2d.$ $a-\frac{3d}{2},\ a-\frac{d}{2},\ a+\frac{d}{2},\ a+\frac{3d}{2}.$

XLII. c. (p. 365).

1. 2, 5, 8, ... **2.** 46, 45, 44, ... **3.** −58, −51, −44, ...

4. $12\frac{3}{4}, 12\frac{1}{2}, 12\frac{1}{4}, \ldots.$ **5.** $\frac{1}{2}(5n^2+13n).$ **6.** $\frac{1}{2}(7n^2+13n).$

7. 19. **8.** 7, 1390. **9.** 17, 23, 29, ...

10. $5\frac{2}{3}, 6\frac{1}{3}, 7, \ldots.$ **11.** 32, 29, 26, ... **12.** $x+\frac{2}{3}y,\ x+\frac{1}{3}y,\ x, \ldots.$

13. 25. **14.** $\frac{5}{12}$. **15.** 2940. **16.** 18, 348.

17. 380. **18.** 2, 4, 6, ... **19.** 8, 12, 16, ..., or 32, 28, 24.

20. 3 shillings. **21.** In 207 days, 4140 miles west of the starting point.

22. $\frac{8a-9b}{6},\ \frac{5a-3b}{3}, \ldots.$ **23.** $n=10.$

24. $3, 2\frac{1}{2}, 2, \ldots.$ **25.** 1, 3, 5, ...

26. $\frac{1}{2}(2n+1)(a+c);\ \frac{a+c}{2}.$ **27.** $\frac{qm-pn+p-q}{m-n};\ d=\frac{p-q}{m-n}.$

XLIII. a. (p. 369).

1. 48. **2.** $\frac{243}{2}$. **3.** $\frac{1}{256}$.

4. 3. **5.** $65535\frac{3}{4}$. **6.** $\frac{a(1-b^{2x})}{1-b^2}$.

7. $\frac{1-c^{30}}{1+c^3}$. **8.** $-\frac{31}{2}(2-\sqrt{2})$. **9.** $\frac{40}{3}(3+\sqrt{3})$.

10. $\frac{5^6-3^6}{2\times 5^4}$. **11.** −1533. **13.** $\frac{463}{81}$.

14. $\frac{133}{27}$. **15.** $\frac{211}{81}(\sqrt{3}-\sqrt{2})$. **16.** $-a\frac{1-(-a)^n}{1+a}$.

17. 189 **18.** $21\frac{1}{2}$. **19.** ±10, 20, ±40.

20. $\pm 27, \frac{27}{2}, \pm\frac{27}{4}$. **21.** $+14, 9\frac{1}{3}, \ldots.$ **22.** ±192, 384, ±768.

23. ±21. **24.** ±1. **25.** ±54. **26.** 1 or 0 according as n is odd or even.

27. 32. **28.** 127. **29.** 1. **30.** 83.

31. −1. **32.** −25. **33.** $3^{n+1}-1$. **34.** a^{n-6}.

35. $\frac{1}{4}$. **36.** 4. **37.** 2^{n-3}. **38.** $a^{n+2}-b$.

XLIII. b. (p. 370).

2. $\frac{1}{2}$. **3.** $\frac{27}{5}$. **4.** 16. **5.** $\frac{128}{7}$. **6.** $\frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}(a+b)}$.

7. $(1-x)^{-2}$. **8.** $(1+x)(1-x)^{-2}$. **9.** $2(2^n-1)-n$.

10. $\frac{8}{9}$. **11.** $\frac{68}{111}$. **12.** $\frac{61}{495}$. **13.** $\frac{199}{825}$.

16. ±14, 28, ±56. **17.** −6, 18. **18.** $\frac{1}{7}$.

20. $3+9+27+\ldots$ or $48-36+27-\ldots$. **21.** The 7th. **23.** 76.

25. $\dfrac{x^{b-2c}(1-x^{nc})}{1-x^c}$. **26.** $2^{n+1}-3$.

27. $\dfrac{1}{2}\left(\dfrac{a}{1-r}\right)^2-\dfrac{1}{2}\cdot\dfrac{a^2}{1-r^2}$. **28.** $\dfrac{64}{9}$, or $\dfrac{64}{63}$.

XLIV. (p. 374).

1. $\frac{4}{11}$, $\frac{1}{3}$, $\frac{4}{13}$. **2.** $\frac{2}{9}$, $\frac{1}{4}$, $\frac{2}{7}$. **3.** $\frac{1}{6}$, $\frac{3}{23}$, $\frac{3}{28}$. **4.** -4.

5. $\frac{1}{39}$. **6.** $-\frac{3}{14}$. **7.** $\frac{6}{5}$, $\frac{3}{2}$, 2, 3.

8. $1\frac{19}{61}$, $1\frac{19}{21}$, $3\frac{11}{23}$. **9.** $\frac{12}{5}$, 3, 4, 6. **10.** 7, $5\frac{8}{11}$, $4\frac{11}{13}$. **11.** $\frac{3}{2}$.

XLV. a. (p. 379).

1. 120. **2.** 220. **3.** 680. **4.** 285. **5.** 650.

6. 4. **7.** 224. **8.** 155. **9.** 174. **10.** 28, 56.

11. 6, 196. **12.** 30. **13.** 123225.

XLV. b. (p. 380).

4. $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$. **5.** $\frac{1}{3}n\,.\,\overline{n+1}\,.\,\overline{n+2}$. **8.** -3, -5, -7, -9.

10. G.P. ratio $\dfrac{c}{a}$. **11.** 6. **12.** $a+nd$.

13. $na^2+n(n-1)ad+\frac{1}{6}n(n-1)(2n-1)d^2$. **14.** 37674.

15. 78^2. **16.** 231. **17.** $\dfrac{n}{6}\{2n^2+3n-5\}$.

18. 5 or 10. **19.** 26th. **20.** $\dfrac{a^2(r^{4n}-1)}{r^4-1}$.

21. $\frac{1}{4}(n-1)n(n+1)(n+2)$. **22.** $n^2(n+1)$.

23. $\frac{1}{3}n(n^2+6n+11)$. **24.** $\frac{1}{6}n(n+1)(2an+a+3b)$.

25. $\dfrac{n}{12}\{2n^2+3n-1\}$. **26.** a, $3a$, $5a$, $7a$, ...

27. 9 or $\frac{1}{9}$. **28.** n^2, $\dfrac{1}{n(n+1)(n+2)}$. **29.** $\frac{5}{2}\left\{1-\dfrac{2^n}{3^n}\right\}$, $\frac{5}{2}$.

30. $\dfrac{36}{11}\left(1-\dfrac{5^{2n}}{6^{2n}}\right)$, $\dfrac{36}{11}$. **31.** $-2n$.

32. $nx^2+n\,.\,\overline{n-1}\,.\,x+\frac{1}{6}n\,.\,\overline{n-1}\,.\,2\overline{n-1}$. **33.** 2805.

35. d. **37.** $\dfrac{1-x^n}{(1-x)^2}-\dfrac{nx^n}{1-x}$. **42.** $\frac{4}{3}(2^{2n}-1)-n(n+1)$.

43. 49, 1. **44.** 2, 4, 8. **45.** $2n-\dfrac{2}{7}$.

46. 3 : 1. **49.** $\dfrac{n}{2}(n+1)$, $\dfrac{n}{6}(n+1)(n+2)$.

50. (i) $\dfrac{n}{3}(n+1)(n+2)$. (ii) $\dfrac{n}{3}(4n^2+6n-1)$. (iii) $\dfrac{n}{4}(n-1)(n-2)(n-3)$.

52. (i) $\dfrac{a-(a+n-1)x^n}{1-x}+\dfrac{x(1-x^{n-1})}{(1-x)^2}$. (ii) $\dfrac{9}{4}-\left(\dfrac{9}{4}+\dfrac{3n}{2}\right)\dfrac{1}{3^n}$.

(iii) $n(n+1)(2n^2+6n+1)-3n$. **53.** $\frac{2}{9}$.

54. (i) $n^2(2n+1)$, $\frac{1}{6}n(n+1)(3n^2+5n+1)$.

(ii) $(3n-2)(3n+1)$, $3n^3+3n^2-2n$.

XLVI. a. (p. 384).

1. $\frac{2}{3}$. **2.** a. **3.** 4, 11, 18. **4.** $\frac{1}{2}$. **6.** $\dfrac{n(n+1)}{2}a+nb$.

XLVI. b. (p. 384).

1. ·4142. **2.** 4 or 2. **4.** 2, 3, 6.

5. 13. **6.** $a^{\frac{1}{2}n.\overline{n+1}}$. **7.** 1·382, −4.

XLVI. c. (p. 384).

4. 3, 9, 15. **5.** a. **6.** $2(2^n-1)-\dfrac{n(n+1)}{2}$. **7.** 70.

XLVI. d. (p. 385).

1. 3, 7, 11, 15. **2.** $2\sqrt{b^2-c^2}$, $c^4x^2-4bc^2x+4b^2=0$. **3.** 20.

4. $9\frac{2}{3}$, $10\frac{1}{3}$; 460. **5.** 18. **6.** $\dfrac{1+e^2}{1-e^2}.h$.

XLVI. e. (p. 385).

1. $x^2+3px+2p^2+q=0$. **2.** A 16, B 40. **3.** 5·82.

4. $\frac{2}{3}n(n+1)(2n+1)$. **5.** $\dfrac{a}{4}, \dfrac{a}{16}, \dfrac{a}{64}\ldots$.

XLVI. f. (p. 386).

2. $2\sqrt{5}:\sqrt{13}$. **4.** $\sqrt{17}$. **5.** 4·919. **7.** 1·309 feet.

XLVI. g. (p. 386).

2. 60°, 108°. **4.** ·6309. **6.** 3·97 inches.

7. $(\frac{1}{4}, 1)$. The line is a tangent there.

XLVI. h. (p. 386).

1. 3 and 4, 4 and 5, 7 and 8, 0 and 1, 0 and 1.

2. $5^{\frac{1}{5}}>6^{\frac{1}{6}}>7^{\frac{1}{7}}$; 2·46. **3.** 12. **4.** $3+2\frac{1}{2}+2+\ldots$.

5. 3·125 sq. in. **6.** 14, 3479. **7.** 5 secs.

XLVI. i. (p. 387).

2. 5 : 24. **3.** 90, 110. **4.** diff. $=\dfrac{a}{3}+\dfrac{b}{2}$. **5.** ±4. **6.** 1, $\frac{1}{4}$.

XLVI. k. (p. 387).

1. 3. 2. 14, 67. 4. $\sqrt{2x-3}+\sqrt{x+2}$. 7. 2, 3, 6.

XLVI. l. (p. 388).

2. $\left\{\frac{n(n+1)}{2}\right\}^2-\left\{\frac{n(n-1)}{2}\right\}^2$. 3. 1·242. 7. $1\frac{1}{2}$ acres.

XLVI. m. (p. 388).

1. £4500, £2700. 2. 38th; 10. 3. $\frac{15}{4}$, 5, $\frac{25}{4}$.

4. $\frac{a}{8}(31+15\sqrt{2})$; $a=2(2-\sqrt{2})$. 5. 4·983. 7. $2\frac{1}{4}$ hrs.

XLVII. a. (p. 393).

1. 1155. 2. 3050. 3. 16215. 4. $41t$. 5. $100e$.
6. ·241. 7. ·27065.... 8. ·54$\dot{1}$. 9. 113·02.
10. 678. 11. 1739. 12. 2518. 13. 1543.
14. $t968$. 15. $t968$. 16. 1360. 17. 781.

XLVII. b. (p. 393).

1. 4155. 2. 100242. 3. $1t8e54$. 4. 222·22, 22·$\dot{6}35\dot{4}$.
5. ·$\dot{3}$. 6. 336532. 7. 26·5. 8. 30·25.
9. 23784. 10. 2653919. 11. $3tt{\cdot}19$.
12. 1024; 605820, 345, 1756. 13. 423. 14. 18·355.
15. 6. 16. 9. 17. The denary number 47. 18. 5.
19. The denary number 17. 20. Use the weights $1+2^2+2^4+2^6$.
21. Put in one scale 2 weights of 1 lb., and 1 each of 3, 3^2, 3^4; (2) put in one scale 3^4+3^3, in the other 3^2+3+1.
22. $3^5+3^4+3^2+1$ in one scale, 3^3+3 in the other.
23. Undetected errors, errors of 9 or a multiple of 9, errors of figures transposed, any error which does not alter the sum of the digits in the result or alters the sum by a multiple of 9.

XLVIII. a. (p. 395).

1. 4. 2. 24. 3. 6. 4. 12. 5. 10. 6. $\frac{n}{n-r}$.
7. 7. 8. $n-6$. 9. $n-r+1$. 10. 12; ${}^{13}P_3\div\underline{|3}$.

XLVIII. b. (p. 396).

1. 840; 504; 30240. 2. 5040. 3. $\underline{|n-1}$. 4. 144.
5. 10. 6. 7. 7. 6. 8. $\frac{\underline{|7}}{\underline{|2}\ \underline{|2}}$; $\frac{\underline{|11}}{\underline{|2}\ \underline{|2}\ \underline{|2}}$. 120.
9. $\underline{|4}=24$. 10. $\frac{1}{2}\underline{|6}=360$. 11. $\underline{|6}=720$. 12. $\underline{|n-1}$.

XLVIII. c. (p. 401).

1. 84, 84, 56, 56, 66. **2.** 9. **3.** 5. **4.** 210, 120.

5. 252, 126, 126. **6.** 11760. **7.** 4080.

8. $\frac{\lfloor 11}{\lfloor 5 \lfloor 3 \lfloor 2}$. **9.** $\frac{\lfloor 13}{\lfloor 3 \lfloor 2 \lfloor 2 \lfloor 2}$. **10.** 1001. **11.** 6.

12. $x=4$. **13.** $\lfloor 8,\ 3 \lfloor 7$. **14.** $\frac{\lfloor 8}{\lfloor 3 \lfloor 3}$, $\frac{\lfloor 6}{\lfloor 3 \lfloor 2}$. **15.** 24.

16. ${}^{30}C_1+{}^{30}C_2+{}^{30}C_3+{}^{30}C_4+{}^{30}C_5$; ${}^{29}C_4$. **17.** 24, 66660.

18. $720 \lfloor 5$. **19.** 28. **20.** 3784 (*one person does not constitute a party*).

21. $\frac{7 \lfloor 89}{\lfloor 4 \lfloor 85}$. **22.** 9; $\frac{\lfloor 15}{\lfloor 6 \lfloor 9}$. **23.** 1440, 480. **24.** 80, 7. **25.** 10.

26. 35^2. **27.** 675. **28.** $\frac{\lfloor 52}{(\lfloor 13)^4}$, $\frac{24 \lfloor 52}{(\lfloor 13)^4}$. **29.** 495.

30. 1001. **31.** 15. **32.** 60. **33.** 9000.

34. $\frac{6 \lfloor 15}{\lfloor 5 \lfloor 5 \lfloor 5}$. **35.** 225. **36.** $np+2$. **37.** $\frac{\lfloor 12}{\lfloor 7 \lfloor 5}$.

38. 113322. **40.** 7560. **42.** 19 in particular case, 18 in general.

43. 72. **44.** 3360, 72.

XLIX. a. (p. 407).

1. $n+1$. **2.** The 6th. **3.** a^{n-5}. **4.** $\frac{n(n-1)}{2}$.

5. $\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}$. **6.** It contains x^r. **7.** $\lfloor r$.

8. Put $-x$ for x, and so make alternate terms negative. **9.** The 9th.

10. 1, 4, 6, 4, 1. **11.** 1, 5, 10, 10, 5, 1. **12.** 1, 6, 15, 20, 15, 6, 1.

13. $\frac{n-r+1}{r}x$. **14.** $\frac{n-r+1}{r} \cdot \frac{x}{a}$. **15.** The 4th.

16. 20. **17.** By putting $x=1$. **19.** 1st, 3rd, 5th, 7th.

20. 9th. **21.** 4th. **22.** 10. **23.** 10.

24. $p+1$. **25.** r. **26.** $r-1$. **27.** $r+2$.

28. (i) $(n-1)$th; (ii) $(n-3)$th; (iii) $(n-r+2)$th; (iv) $(n-r+1)$th.

29. (i) $(n-3)$th; (ii) $(n-6)$th: (iii) $(n-r+1)$th; (iv) $(n-r)$th.

30. (i) $(n+1)$th; (ii) $\frac{\lfloor 2n}{(\lfloor n)^2}a^n x^n$; (iii) $\frac{\lfloor 2n}{\lfloor n-1 \lfloor n+1}a^{n-1}x^{n+1}$;

(iv) $\frac{\lfloor 2n}{\lfloor n+1 \lfloor n-1}a^{n+1}x^{n-1}$.

31. ${}^nC_3a^3x^{n-3}+{}^nC_2a^2x^{n-2}+{}^nC_1ax^{n-1}+x^n$.

32. (i) ${}^{25}C_{r-1}z^{r-1}x^{26-r}$; (ii) ${}^{25}C_r z^r x^{25-r}$; (iii) ${}^{25}C_{r-2}z^{r-2}x^{27-r}$;
(iv) ${}^{25}C_{r-1}z^{26-r}x^{r-1}$; (v) ${}^{25}C_r z^{25-r}x^r$.

XLIX. b. (p. 411).

1. $1+5x+10x^2+10x^3+5x^4+x^5$.
2. $a^7+7a^6x+21a^5x^2+35a^4x^3+35a^3x^4+21a^2x^5+7ax^6+x^7$.
3. $a^{12}-6a^{10}x^2+15a^8x^4-20a^6x^6+15a^4x^8-6a^2x^{10}+x^{12}$.
4. $a^8b^8-4a^7b^7x+7a^6b^6x^2-7a^5b^5x^3+\frac{35}{8}a^4b^4x^4-\frac{7}{4}a^3b^3x^5+\frac{7}{16}a^2b^2x^6-\frac{1}{16}abx^7+\frac{1}{256}x^8$.
5. $x^8+8x^6+28x^4+56x^2+70+\frac{56}{x^2}+\frac{28}{x^4}+\frac{8}{x^6}+\frac{1}{x^8}$.
6. $a^7x^7-7a^6bx^5+21a^5b^2x^3-35a^4b^3x+\frac{35a^3b^4}{x}-\frac{21a^2b^5}{x^3}+\frac{7ab^6}{x^5}-\frac{b^7}{x^7}$.
7. $243x^5+810x^4y+1080x^3y^2+720x^2y^3+240xy^4+32y^5$.
8. $2(a^8+56a^6b^2+280a^4b^4+224a^2b^6+16b^8)$.
9. $210x^4$.
10. $462a^{12}x^{10}$.
11. $-1365\times3^4y^{11}$.
12. 924.
13. $-462\times2^6x+462\times\frac{2^5}{x}$.
14. $\frac{2n(2n-1)\ldots(2n-r+1)}{\lfloor r}x^r$; $(-1)^r\cdot\frac{n(n-1)\ldots(n-r+1)}{\lfloor r}\cdot3^{n-r}2^rx^r$.
17. $\frac{n}{2}-1$.
18. $\frac{\lfloor n}{\lfloor r\,\lfloor n-r}2^rx^r$.
19. $(-1)^r\frac{\lfloor 2n}{r\,\lfloor 2n-r}\cdot3^{2n-r}\cdot5^rx^r$.
20. $(-1)^r\frac{\lfloor n}{r\,\lfloor n-r}3^ra^{2n-2r}b^rx^r$.
21. $\frac{\lfloor n}{\lfloor r\,\lfloor n-r}x^{n-2r}$.
22. $(-1)^n\frac{\lfloor 2n}{\lfloor n\,\lfloor n}$.
23. $4032x^5y^4$, $2016x^4y^5$.
24. 80.
25. 21.
26. 252×2^5.
28. $a^9-5a^8x+8a^7x^2-14a^5x^4+14a^4x^5-8a^2x^7+5ax^8-x^9$.
29. $x^8-2x^6-2x^4+6x^2-\frac{6}{x^2}+\frac{2}{x^4}+\frac{2}{x^6}-\frac{1}{x^8}$.
30. $1792\sqrt{3}$.
33. $-\frac{11x}{5}$.
35. ·9988.
36. ${}^{14}C_7$.
37. The 6th = the 7th = 7.
38. 6th $=42\times12^5$.
39. 5th $=240\times9^4$.
40. 3rd $=\frac{15}{4}$.
41. 4th $=\frac{35}{432}$.
42. 5^n.
43. 1.
44. $\left(\frac{1+2x}{1+x}\right)^n$.
45. ${}^{20}C_{10}$.
46. 8.
47. -134.
55. $\frac{(1+x)^n+(1-x)^n}{2}$.

XLIX. c. (p. 418).

1. $a^{-5}+10a^{-6}x+60a^{-7}x^2+280a^{-8}x^3$.
2. $1+x-\frac{1}{2}x^2+\frac{1}{2}x^3$.
3. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3$.
4. $1-\frac{10}{3}x+\frac{20}{3}x^2-\frac{220}{27}x^3$.
5. $a-\frac{1}{2}a^{-1}x^2-\frac{1}{8}a^{-3}x^4-\frac{1}{16}a^{-5}x^6$.
6. $1+\frac{1}{4}x-\frac{1}{32}x^2+\frac{1}{128}x^3$.
7. $1+\frac{1}{3}x^2+\frac{2}{9}x^4+\frac{14}{81}x^6$.
8. $1+x+2x^2+\frac{14}{3}x^3$.

9. $a^{-2}-a^{-4}b^2+a^{-6}b^4-a^{-8}b^6$. **10.** $\frac{x^3}{16}$.

11. $715(2b)^4(3a)^{-14}$. **12.** $-\frac{5}{1024}b^{18}$.

13. $(-1)^r\frac{n.(n+1)(n+2)\dots(n+r-1)}{\lfloor r}x^{-r-n}a^r$. **14.** $\frac{(r+1)(r+2)}{1.2}x^r$.

15. $\frac{(r+1)(r+2)(r+3)(r+4)}{\lfloor 4}x^r$. **16.** $\frac{(r+1)(r+2)(r+3)(r+4)}{\lfloor 4}\frac{x^r}{2^{r+5}}$.

17. $(-1)^r\frac{2.5.8\dots(3r-1)}{3^r\lfloor r}x^r$.

19. $1+\frac{n}{x}+\frac{n.(n+1)}{\lfloor 2}\frac{1}{x^2}+\frac{n(n+1)(n+2)}{\lfloor 3}\cdot\frac{1}{x^3}+\dots$.

$T_{n+1}=\frac{n.(n+1)(n+2)\dots(2n-1)}{\lfloor n}\cdot\frac{1}{x^n}$.

$=\frac{\lfloor 2n-1}{\lfloor n\ \lfloor n-1}.n^n$ when $x=\frac{1}{n}$.

20. $\frac{\lfloor n}{\lfloor\frac{1}{2}(n+r)\ \lfloor\frac{1}{2}(n-r)}$. **21.** -105. **22.** $5r+3$.

23. $2r^2-6r+1$. **24.** 46. **25.** $4r+1$.

26. $(r+1)(r+3)$. **28.** $(-1)^2\cdot\frac{\lfloor 2n}{\lfloor r\ \lfloor 2n-r}$.

29. 4th. **30.** All except 2. **31.** The 2nd.

32. 7th and 8th equal. **33.** 2nd = 3rd = greatest. **34.** 3rd.

XLIX. d. (p. 422).

1. (i) 1·10. (ii) 1·16. (iii) 1·22. (iv) 1·006. (v) 1·170. (vi) 1·01206. (vii) ·99700. (viii) ·9224.

2. 1030·301, 1060·90. **3.** 10406·04, 10820·12. **4.** 970·299, 940·89.

5. (i) ·31833, $\frac{1}{\pi}=$·31831. (ii) $\pi^2=9\cdot87$, $\frac{1}{\pi^2}=(10-\cdot13)^{-1}$.

6. (i) 1·0005. (ii) 1·00020. (iii) 1·00025005. (iv) 1·001.

7. (i) 10·017. (ii) 9·949875. (iii) 1·98734.

8. (i) 7·98957. (ii) 3·03659. (iii) 4·99936.

9. ·2 per cent. **10.** About ·0001. **11.** $1+\frac{1}{2}x-\frac{1}{8}x^2$.

12. 1·015. **13.** 3 places.

14. $1-5x+5^2x^2-\dots$, $\frac{1}{5x}\left\{1-\frac{1}{5x}+\frac{1}{5^2x^2}-\dots\right\}$. **16.** 4·850.

17. ·03 per cent. **18.** 17 approx. **19.** $28x^2$. **26.** $\frac{9}{4}$.

27. $\frac{125}{27}$. **29.** $\frac{n.\overline{n+1}.\overline{n+2}.\overline{n+3}}{4}$. **30.** $\frac{1.3.5\dots\overline{2n-3}}{2^{n-1}\lfloor n-1}$.

31. $\frac{5.7.9\dots\overline{2n+1}}{2^{n-1}\lfloor n-1}$. **32.** n^2. **34.** $\frac{1.4.7\dots\overline{3r-2}}{\lfloor r}x^{-6r-1}$.

35. 2·25 grms. **36.** 1.

L. a. (p. 425).

1. $x+y+z$. **2.** 1.

3. $c^2+(m+n)(2q-p^2)x+(m^2+n^2)q^2+mn(p^4-4p^2q+2q^2)=0$.

4. 0, -5. **6.** $2\cdot4950$, $\bar{1}\cdot3495$, $\bar{8}\cdot7760$. **7.** $\cdot25$, 1.

L. b. (p. 425).

1. $\sqrt{5}+\sqrt{3}$. **2.** 15120. **3.** 30 miles.

4. $\cdot23456$. **5.** $\frac{1}{6}$. **6.** (i) $a=-b$. (ii) $2a^2-3ab+2b^2=0$.

L. c. (p. 426).

2. 12. **3.** 24. **4.** 11 years. **5.** $-{}^{15}C_6 \cdot \frac{1}{x^3}$. **6.** $\cdot28854$.

L. d. (p. 426).

1. a^2. **2.** $\bar{1}\cdot738$. **3.** 28·8 in.

4. 4. **5.** $a^{-1}+2-b^{-2}$. **6.** $\frac{(r+1)(r+2)}{2} \cdot \frac{x^r}{3^{2r+3}}$.

7. 9·99998999998.

L. e. (p. 426).

1. $\frac{3}{2}$, -1, 2, $-\frac{3}{2}$. **2.** 52. **4.** 121234·1104.

6. $3 \cdot 2^{-\frac{2}{2}\frac{7}{2}} x^{\frac{11}{2}}$, $\frac{1 \cdot 3 \cdot 5 \ldots (2r-1)}{2^r \lfloor r}$. **7.** 18.

L. f. (p. 427).

1. $3x^2+4x-5$. **2.** $(a-b)^2+(b-c)^2+(c-a)^2$. **3.** 1·153, 1, 23·11.

6. $\frac{1}{(x-y)(x^2-xy+y^2)}$. **7.** 16 years.

L. g. (p. 427).

1. $x=2$, $-\frac{11}{8}$, -2, $\frac{11}{8}$,
$y=1$, $-\frac{19}{8}$, -1, $\frac{19}{8}$.

2. 2. **3.** 20.

4. 7. **5.** 63.

6. $3+\sqrt{2}(=4\cdot414)$.

L. h. (p. 428).

1. A. P. $-\frac{1}{3+2\sqrt{2}}$.

2. $\sqrt{2x+1}+\sqrt{x-4}$.

3. $\frac{2}{1}\frac{2}{7}$.

5. $\frac{\lfloor 10}{\lfloor 3 \lfloor 7}$; $\frac{2 \lfloor 10}{\underline{3} \lfloor 7}$.

6. (i) The 11th and the 12th. (ii) The 13th.

7. $\frac{(r+1)(3r+2)}{2}$.

L. k. (p. 428).

2. $(a+b)(a^{\frac{1}{3}}+b^{\frac{1}{3}})/a^{\frac{2}{3}}b^{\frac{2}{3}}$. 3. $1, -\frac{3}{5}$. 4. ·785....
6. The 5th. 7. $\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$.

L. l. (p. 428).

2. $x=\frac{1}{4}$, $y=-\frac{1}{2}$. 3. 144. 4. $\bar{3}$·123440.
5. 8190 nearly. 7. $\left(\frac{\lfloor 12}{\lfloor 5 \lfloor 7}\right)^2=627264$.

L. m. (p. 429).

2. $\frac{4}{3}$. 3. $1+\frac{1}{2}\sqrt{2}$. 4. 0. 5. $4\frac{1}{2}$, 4. 7. $\log_a 2$, -1.

L. n. (p. 429).

3. $a^3-3ab+2c=0$. 4. $x=\pm 2, \pm 3$; $y=\pm 3, \pm 2$.
5. ${}^{14}C_8$. 6. $\frac{1}{6}(r+1)(r+2)(2r+3)$. 7. The 4th and 5th.

L. p. (p. 430).

3. 469. 5. (i) 54321; (ii) 7465. 6. ·9895.
7. $1+4x+10x^2+20x^3+35x^4$; $\frac{1}{6}(r+4)(r+5)(r+6)x^{r+3}$.

L. q. (p. 430).

1. £50. 2. $\frac{2(2a+d)}{3d}$ or $\frac{2a+d}{3d}$. 3. $2^{\frac{1}{3}}=1{\cdot}260$.
5. $\frac{a}{m}$. 6. $\sqrt{2}$. 7. $\frac{2n(2n+1)(2n+2)\ldots(2n+r-1)}{\lfloor r}$.

L. r. (p. 430).

1. $\log 2$. 3. $\frac{1}{2}a(3b^2-a^2)$. 4. $x=\frac{1}{5}$.
6. 3, 4, 5, 6, 7, or $-6, -\frac{1}{2}, 5, 10\frac{1}{2}, 16$. 7. 35700.

L. s. (p. 431).

1. $\frac{n}{2}(4n^2+11n+9)$. 2. $\lfloor 52/(\lfloor 13)^4$. 3. 3·45. 4. 7·9896.
5. For the motion, 300; against it, 260. 6. 14.
7. Coeff. of x^{r-1} in $(1-x)^{-6}$.

L. t. (p. 431).

1. 271. 2. $1+3^4+3^6$ in one scale, 3 lbs. in the other. 6. ·01266.

LI. a. (p. 432).

1. $2\frac{3}{11}$ %. 2. 5 years. 3. £4. 9*s*. 5·29*d*. 4. £98. 10*s*.
5. £23. 19*s*. 2·4*d*. 6. £1014. 7. £159.
8. £59. 19*s*. 10*d*. to the nearest penny.
9. 14·198 yrs., *i.e.* 14 yrs. 2·38 months. 10. 17·706 yrs.
11. £70. 9*s*. 5*d*. to the nearest penny. 12. 25·294 yrs.
14. $4\frac{1}{2}$ %. 15. $3\frac{1}{3}$ yrs. 16. £554. 10*s*. 17. 120 days.
18. £8042. 19. 4 %. 20. $3\frac{3}{4}$ yrs. 21. £750. 16*s*. 8*d*.
22. 37·27 yrs. 23. 21 : 20. 24. £1030 ; 4 %.
25. $1 + \cdot15 + \cdot009 + \cdot00027 + \cdot00000405 + \cdot0000000243$; £15. 18*s*. $6\frac{1}{2}$*d*.
27. £8318.

LI. b. (p. 435).

1. £14000. 2. £7683. 14*s*. 11*d*. 3. 4 %.
4. $5\frac{1}{2}$ % ; $18\frac{2}{11}$. 5. £873. 14*s*. 3*d*., £2171. 8*s*. 6*d*. 6. 4·8 %.
7. £162. 18*s*. 8. $22\frac{2}{9}$, 20. 9. £512. 14*s*.
10. £1057. 7*s*. 11. £1495. 18*s*. to nearest shilling. 12. £8000.
13. £112. 16*s*. 14. £817. 4*s*.

LII. (p. 441).

2. e^{-2}. 3. $2\log_e 2 - \log_e 3$. 6. $3x - \frac{1}{2}(1+2^2)x^2 + \frac{1}{3}(1+2^3)x^3 - \ldots$
7. $x + \frac{1}{2}(3^2-2^2)x^2 + \frac{1}{3}(3^3-2^3)x^3 + \ldots$. 8. ·0392.
9. $x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 - \ldots - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \ldots$. 10. $2\log\left(x + \frac{1}{x}\right)$.
11. 1·098612. 12. 2·302585 ; ·43429. 13. 1. 14. $\frac{1}{2}\log\frac{x+1}{x-1}$.
15. $2e$. 16. ·00100000008. 19. 8·082.

LIII. a. (p. 444).

1. $\frac{n(4n^2-1)}{3}$. 2. $\frac{n(n+1)(n+2)}{3}$. 3. $\frac{n(n+1)(4n-1)}{3}$.
4. $(4pr-q^2)^2 = 64r^3$. 5. $pq = r$. 6. $a = b = 3$.
7. $a = -23$, $b = 18$. 8. $c = 3$. 10. $a = 7$, $b = 3$, $c = -7$.
11. $a = -\frac{1}{2}$, $b = -4$, $c = \frac{9}{2}$. 12. $a = 5$, $b = -2$, $c = 2$, $d = -3$, $e = 1$.
13. $\frac{1}{2} - \frac{3}{4}x + \frac{9}{8}x^2 - \frac{27}{16}x^3 + \ldots$. 14. $1 + x + 4x^2 + 10x^3 + \ldots$.
15. $1 + 2x + 7x^2 + 26x^3 + \ldots$. 16. $1 + 3x + 10x^2 + 36x^3 +$
17. $\frac{1-x}{1-x+x^2}$. 18. $\frac{1}{(1-x)^2}$. 19. $\frac{2-5x}{1-5x+4x^2}$. 20. $\frac{1+2x}{1-x-x^2}$.

LIII. b. (p. 447).

1. $\frac{1}{x+1}+\frac{1}{x-1}$.
2. $\frac{2}{x-2}-\frac{1}{x-3}$.
3. $\frac{1}{x-2}-\frac{1}{x+2}$.
4. $\frac{1}{2(x+1)}-\frac{1}{2(x+3)}$.
5. $\frac{1}{1-x}-\frac{3}{2-x}$.
6. $\frac{1}{x+1}-\frac{2}{x+2}+\frac{1}{x+3}$.
7. $\frac{1}{1+x}+\frac{4}{1+3x}$.
8. $\frac{3}{x}-\frac{2}{2x-1}-\frac{4}{2x+1}$.
9. $\frac{18}{x-3}-\frac{10}{x-2}-\frac{2}{x-1}$.
10. $\frac{3}{2(x-2)}-\frac{1}{x+2}-\frac{x+10}{2(x^2+4)}$.
11. $\frac{1}{2(x-1)}-\frac{1}{2(x+1)}-\frac{1}{x^2}$.
12. $\frac{1}{x-2}-\frac{1}{x+2}-\frac{4}{(x+2)^2}$.
13. $\frac{1}{6(x+1)}-\frac{1}{2(x-1)}+\frac{4}{3(x-2)}$.
14. $x+\frac{4}{3(x-1)}-\frac{1}{3(x+2)}$.
15. $-\frac{3}{x+2}+\frac{3}{x+3}+\frac{4}{(x+3)^2}$.
16. $\frac{4}{x+2}+\frac{5}{(x+2)^2}-\frac{4x-3}{x^2+1}$.
17. $1+\frac{1}{x}+\frac{2x+4}{x^2+x+1}$.
18. $\frac{1}{1-3x}-\frac{1}{1-2x}$; general term $(3^n-2^n)x^n$.
19. $\frac{2}{1-4x}-\frac{1}{1+3x}$; $\{2\cdot 4^n-(-1)^n 3^n\}x^n$.
20. $\frac{1}{x-4}-\frac{1}{x+1}$; $\left\{-\frac{1}{4^{n+1}}+(-1)^{n+1}\right\}x^n$.
21. $\frac{2}{1-2x}-\frac{9}{1-3x}+\frac{8}{1-4x}$; $(2^{n+1}-3^{n+2}+2^{2n+3})x^n$.
22. $\frac{a^2}{(a-b)(a-c)}\cdot\frac{1}{1-ax}+\frac{b^2}{(b-c)(b-a)}\cdot\frac{1}{1-bx}+\frac{c^2}{(c-a)(c-b)}\cdot\frac{1}{1-cx}$; $-(b-c)(c-a)(a-b)$; $-(b-c)(c-a)(a-b)(a+b+c)$.
23. $\frac{3}{2-3x}-\frac{5}{1+2x}$; $\left\{\frac{3^{n+1}}{2^{n+1}}+(-1)^{n+1}\cdot 5^n\right\}x^n$.
24. $4x+12+\frac{32}{x-2}-\frac{4}{x-1}$; $(4-2^{4-n})x^n$, n being >1.
25. $\frac{5}{3(x-1)}+\frac{1}{3(x+2)}$; $\left\{-\frac{5}{3}+\frac{(-1)^n}{6}\cdot\frac{1}{2^n}\right\}x^n$.
26. $\frac{2}{x-2}-\frac{2}{x-1}-\frac{2}{(x-1)^2}+\frac{1}{(x-1)^3}$.
27. $\frac{2}{3(x-1)}+\frac{x-1}{3(x^2+x+1)}$.

LIV. a. (p. 449).

1. 3·1416.
2. 4 %; £3. 15*s*. 6*d*.
3. $x=2y$, 5, $-3\cdot5$.
4. £175.
5. 2·08.
6. 1·2.
7. $a=-5$, $b=\cdot9$; £172, £145, £92.
9. 7.
10. 2 hrs. 32 min. after the man leaves A, nearly 8·9 miles from A.

LIV. b. (p. 449).

2. 7, 13.
3. $h=-14$, $k=12$.
5. 3·16, 2, 3·1623, ·0632.
6. $1+x+x^2+x^3+\ldots$.
7. $x=5$.
8. $4^{-\frac{5}{2}}$.
9. 12; 10 miles an hour.
10. $m=1$ or -7.

LIV. c. (p. 450).

1. $(-1, 2)$, $(0, -2)$, $-\frac{1}{2}$. **2.** ·29053. **3.** $2^{\frac{1}{3}}=1\cdot2599$, $2^{1\frac{7}{2}}=1\cdot4983$.
4. 1·022. **5.** 8·1. **6.** 0, a^2+pa+q. **7.** 3, -5.
8. Parts $\frac{1}{2}(a\pm\sqrt{2b^2-a^2})$, 2·71 or 1·29. **9.** $x^{kp+lq-mr}$.
10. 4, 16, 256, 65536.

LIV. d. (p. 451).

1. $\frac{3}{4}\frac{2}{5}$. **2.** 24395. **4.** The 14th.
5. 1·627. **6.** 143·3. **7.** $(n+1)^2$.
8. 3·13. **9.** x is between 0 and 2. **10.** 16.

LIV. e. (p. 451).

1. $7\sqrt{3}$ $(=12\cdot12)$. **2.** $\sqrt{a(b-x)}\cdot\sqrt{b(a-x)}$. **4.** 4, $3\frac{1}{2}$ miles an hour.
7. 2·6. **10.** (1, 1), (2, 2).

LIV. f. (p. 452).

1. 45, 36 miles an hour. **2.** 8. **3.** 76.
4. 1·5007. **5.** (9, 6), $(-6, -4)$. **6.** 6, $4\frac{1}{2}$.
7. $\frac{\lfloor 20}{\lfloor 5\,\lfloor 15}$, $\frac{\lfloor 20}{(\lfloor 5)^4}$. **8.** $\frac{an(n+1)}{2}+2^n-1$.
9. $a=3$, $b=-3$ or $-\frac{1}{3}$, $c=-\frac{1}{3}$ or -3.

LIV. g. (p. 452).

1. p/q. **2.** (i) 8·116. (ii) 2·477. **3.** $\sqrt{10}$ $(=3\cdot162)$.
4. 5. **6.** $a=-16$, $b=-106$, $c=450$. **7.** A, B, C.
8. ·00000000000106215728. **9.** $\log 2+\frac{x}{2}-\frac{5x^2}{8}+\frac{7x^3}{24}-\ldots$.

LIV. h. (p. 453).

2. b numerically $<ac$; $\frac{a^3c}{b(ac-b)}$. **3.** 5940. **4.** $-\frac{273}{4}$.
5. $(a-b+c)x^2-2(a-c)x+a+b+c=0$. **6.** ·6309.
7. 8th term $=-5049x^7/5^8$. **8.** $\frac{1}{b-a}\left(\frac{1}{a-x}-\frac{1}{b-x}\right)$.
10. 12 miles an hour; £9. 7*s*. 6*d*.

LIV. k. (p. 453).

1. 1·15, 3·68. **2.** $b=a^2+d$, $c=ad$. **3.** 259700.
4. 126. **6.** (3, 6). **7.** The 6th and 5th; also the 25th and 26th.
8. 785·4. **10.** $\log_e 1\cdot2$.

LIV. l. (p. 454).

2. $\frac{a^2+b^2}{2ab}$. **3.** $\frac{\lfloor 17}{\lfloor 10\,\lfloor 7}-\frac{8\,\lfloor 13}{\lfloor 6\,\lfloor 7}+28\times36$. **4.** $\frac{1}{1\frac{1}{2}}$.
6. 2, 6, 10, 14. **7.** n^2. **8.** $x=a$, $y=-b$. **10.** 11.

LIV. m. (p. 454).

1. 2 miles; 24 minutes after the tricycle passed. **2.** a, $-\frac{a}{2}\pm\frac{1}{2}\sqrt{16-3a^2}$.

5. Scale 7; 1313. **6.** 1.

8. $\frac{1}{6}n(n+1)(2n+7)$. **10.** $\frac{1}{2(x^2+1)}+\frac{1}{2(x-1)^2}$.

LIV. n. (p. 455).

1. $2\frac{1}{2\,0}$, $2\frac{1}{1\,0}$, ...; $\frac{1\,2\,0}{5\,9}$, $\frac{6\,0}{2\,9}$, ...

4. $\frac{1}{x-a+b}+\frac{1}{x-a-b}-\frac{2}{x-a}$.

5. $b=3a^2$, $c=a^3$.

6. $1+2x+3x^2+\ldots+(n+1)x^n+\ldots$

$1+3x+6x^2+10x^3+\ldots+\frac{1}{2}(n+1)(n+2)x^n+\ldots$.

$1+4x+10x^2+20x^3+\ldots+\frac{1}{6}(n+1)(n+2)(n+3)x^n+\ldots$.

7. 534. **8.** $d^2=h(2r+h)=2rh$ approximately.

9. To 5 places. **10.** 4198·27 feet.

LIV. p. (p. 456).

2. A 16, B 28. **3.** $(2x-y+1)(x+3y-3)=0$. Two straight lines.

5. $\lfloor n$. **6.** $x=y\log_e b$. **7.** 4·44.

LIV. q. (p. 456).

2. 3 : 5. **3.** 143700 oz. roughly (by 4 figure logarithms).

5. $\frac{1}{x}-\frac{1}{2}\cdot\frac{1}{x^3}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{x^5}-\ldots$. **6.** $y(y^2-3)$, $y(y^4-5y^2+5)$.

7. 6 cwt. **8.** n^2+4n+3, $\frac{n}{6}(2n^2+15n+31)$.

9. $\log 3+x\left(1+\frac{1}{3}\right)-\frac{x^2}{2}\left(1+\frac{1}{9}\right)+\frac{x^3}{3}\left(1+\frac{1}{27}\right)-\ldots$.

LIV. r. (p. 457).

1. $\frac{1}{9}\left(\frac{x^2+y^2}{12}+\frac{145xy}{144}\right)$; 48. **2.** roots $-\frac{\sqrt{5}+1}{2}$, $\frac{\sqrt{5}+3}{2}$.

4. 1. **5.** A's time : B's time = 4 : 7. **6.** 6·24.

7. 41, 40, 9. **8.** 56. **9.** 32.

LIV. s. (p. 457).

1. a. **2.** £34. 1*s*. 9*d*. **3.** 3*s*97*e*.

5. 3·262 feet. **6.** 1·99 inch. **7.** $(\sqrt{3}+1)^{\frac{1}{2}}$, 1·653.

8. $72\lfloor 8$, $56\lfloor 7$. **9.** A 3, B $3\frac{1}{2}$ miles an hour.

LIV. t. (p. 458).

1. 1·2 in. **2.** 5. **3.** 13 : 11 at the beginning, 137293 : 115907 at the end.

5. $T_{12}=-4368a^{-6}$; middle term 12870. **6.** 32·23. **7.** 5·03968.

8. 12 years 16 days. **9.** About 5·77 ft. per second. **10.** $2y$.

LIV. u. (p. 459).

3. $x=6$. **4.** 120·4 **5.** 127.

6. $\frac{1}{2-x}-\frac{1}{3-x}$. (i) $\frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{2^2}+\ldots\right)-\frac{1}{3}\left(1+\frac{x}{3}+\frac{x^2}{3^2}+\ldots\right)$;
(ii) $\frac{1}{x}\left(1+\frac{3}{x}+\frac{3^2}{x^2}+\ldots\right)-\frac{1}{x}\left(1+\frac{2}{x}+\frac{2^2}{x^2}+\ldots\right)$.

7. 19044, sq. rt. 138, 255 = sq. rt. in scale 7. **8.** $n(n+1)-2^{n+1}+2$.

9. 14 digits, first four digits 6756. **10.** $-\frac{3}{4r}$.

LIV. v. (p. 459).

1. 540, 450. **2.** $\frac{4d^4-(a^2-b^2)^2}{4(a^3-ab^2+c^3)}$. **3.** 11. **4.** The 38th, 10.

5. £51. **6.** $\binom{5}{3}^{\frac{7}{2}}+\frac{7}{2}\cdot\binom{5}{3}^{\frac{5}{2}}x+\frac{7\,.\,5}{2\,.\,4}\cdot\binom{5}{3}^{\frac{3}{2}}x^2+\ldots$; the 2nd.

7. $\frac{1}{2}(e^x-e^{-x})$. **8.** 3. **9.** 4*s.* in the £; land : personality = 2 : 3.

LIV. w. (p. 460).

1. x^{pq}, 15. **2.** 10, 24, 26 inches. **4.** 20; $n(n+1)=2\log p/\log a$.

5. 21, 6. **6.** $\frac{9x+13}{5(x^2+1)}+\frac{21}{5(x-2)}$. **7.** £80, £45.

8. $-6435x+6435\,.\,\frac{1}{x}$; 2. **10.** $2e$.

LIV. x. (p. 460).

1. $4a^{\frac{1}{2}}b^{\frac{1}{6}}-5a^{\frac{1}{3}}b^{\frac{1}{3}}+4a^{\frac{1}{6}}b^{\frac{1}{2}}$. **2.** $6\sqrt{3}+5\sqrt{2}$, 17·463.

3. $9(61a-20b)$. **5.** 1·06. **6.** 3·314.

7. (i) $\lfloor\!\underline{6}\,\lfloor\!\underline{3}$. (ii) $120\,\lfloor\!\underline{5}$. **9.** 12.

LIV. y. (p. 461).

1. $\frac{n-1}{n-2}$; when $n=3$. **2.** $12\frac{1}{2}$ miles an hour. **3.** $\frac{\lfloor 12}{\lfloor\!\underline{4}\,\lfloor\!\underline{4}\,\lfloor\!\underline{4}}$.

4. −285. **5.** 62·48. **6.** 8·082. **7.** 10.

8. <1 or >3. **9.** $\frac{(-1)^r}{\lfloor r}(a+br)$. **10.** Just over $23\frac{1}{2}$.

LIV. z. (p. 461).

3. 3·2102. **4.** $(n+2)^3-1$, -1. **5.** 8.

6. Scale 6. **7.** $\frac{(n-1)n(n+1)(n+2)(n+3)}{\lfloor\!\underline{5}}$. **9.** $2n^2(n+1)^2-n$.

LIV. aa. (p. 462).

2. 3808. **4.** 4 days. **7.** 12, 9. **8.** $a:a+x$.

9. 7 : 5. **10.** 5·275 feet.

LIV. bb. (p. 462).

1. 1. **2.** $1+\sqrt{2}$. **3.** 1022634.

4. 2·99753. **6.** $\left\{3(-1)^n-\frac{3^n}{2^{n+1}}\right\}x^n$. **7.** 1·242.

8. $x=\frac{10}{3}$, 5, 10; $y=\frac{16}{3}$, 7, 12. No finite values when $b=1$.

9. 16·7. **10.** 3,950,000.

LIV. cc. (p. 463).

2. 1·473, ·8736. **3.** a not >3 or <1. **5.** $\frac{(1-x^{2n})^2}{1-x^2}$.

6. Between 3 and 4. **8.** 119, 161, 213.

9. 14780, 14340, 13530; 14335 c. ft. **10.** £3675, £2812. 10s., £2870.

LIV. dd. (p. 464).

1. 7·48, 5·98, 5, 4·32. **2.** $3\pm\sqrt{n+1}$, 0, 3, 8, 15. **3.** Between 3 and -1.

4. $y=\frac{10x}{3}-\frac{16}{3x}$. **6.** 1·722, ·0198.

7. 11103·84, 10958·4, 9606·24, 7518·24. **8.** $\frac{2}{x+3}-\frac{1}{x-5}$.

9. 40, 40·401, 40·04001; 40·01, 40·1; 40. **10.** $a=2·2$, $b=·11$, 4·4.

LIV. ee. (p. 465).

2. 1·023. **3.** x must lie between 0 and ·21133 or between ·78867 and unity.

4. -3, -1, 6, -3. **5.** 3·339, 1·930, 1768000, 11·03. **6.** $6>x>1$.

7. 21 horses. **8.** 32 approximately. **9.** He would give away £16.

10. $\frac{2}{x-4}+\frac{1}{x+1}$. **11.** 308, 234 sq. inches. 33930 c. inches.

12. A 2·79, B 2·4; $11\frac{1}{2}$; 4·2, 1·505.

Appendix. (p. 467).

1. $-2·56$, 1, 1·56. **2.** -1, -3, 2.

3. One real root 1; the others imaginary. **4.** 3·45, $-0·75$, $-2·7$.

5. 1·6, $-2·9$, 4·3. **6.** $-1·43$; imaginary.

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